Fourier expansion of a plane wave

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Consider a plane wave propagating with an angle of θ, i.e.,

$$P(r,\phi) = e^{-ikr\cos(\phi-\theta)}. \tag{1}$$

In the above, (r,ϕ) is the polar coordinates and $e^{i\omega t}$ is assumed. We see that the above function is periodic in $\phi-\theta \equiv \alpha$; thus, we can expand the function as a Fourier series, i.e.,

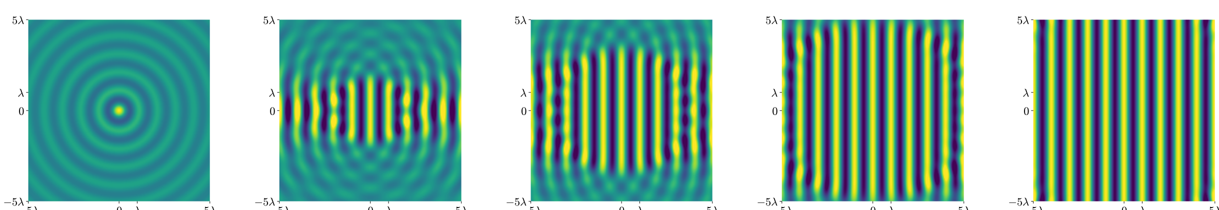
$$P(r,\phi) = e^{-ikr\cos\alpha} = \sum_{n\in\mathbb{Z}} c_n e^{in\alpha}. \tag{2}$$

The corresponding Fourier coefficients c_n can be calculated by

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikr\cos\alpha} e^{-in\alpha} d\alpha \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikr\cos\alpha}\left[\cos(n\alpha) - i\sin(n\alpha)\right] d\alpha \quad (\Leftarrow \text{symmetry}) \\
&= \frac{1}{\pi}\int_{0}^{\pi} e^{-ikr\cos\alpha}\cos(n\alpha)\, d\alpha \quad (\Leftarrow \pi-\tau=\alpha) \\
&= \frac{(-1)^n}{\pi}\int_{0}^{\pi} e^{ikr\cos\tau}\cos(n\tau)\, d\tau \quad \left(\Leftarrow J_n(x) = \frac{i^{-n}}{\pi}\int_0^{\pi} e^{ix\cos\tau}\cos(n\tau)\, d\tau\right) \\
&= i^{-n} J_n(kr).
\end{aligned} \tag{3}$$

Thus, the Fourier series expansion of a plane wave is written in terms of Bessel functions as[1]

$$P(r,\phi) = e^{-ikr\cos(\phi-\theta)} = \sum_{n\in\mathbb{Z}} i^{-n} J_n(kr)\, e^{in(\phi-\theta)}. \tag{4}$$



(a) $N = 0$ (b) $N = 10$ (c) $N = 20$ (d) $N = 30$ (e) $N = 40$

Figure 1: Finite summation of the Fourier series (4); $n \in [-N, N] \subset \mathbb{Z}$; $\theta = 0$; real part

[1] For $e^{-i\omega t}$, we may define $P'(r,\phi) = e^{ikr\cos(\phi-\theta)} = \sum_{n\in\mathbb{Z}} i^n J_n(kr)\, e^{in(\phi-\theta)}$.