Periodically supported beam

Heedong Goh

December 12, 2020

Example 0.1 (Periodically supported beam) *Consider an infinitely long bridge with periodically installed supports, where each span has a length of L. A point load p is applied at the center of a span. We are interested in finding the slope angles θ_l and θ_r at the supports adjacent to the load.*

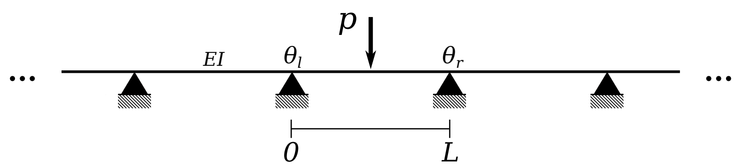

Figure 1: Periodically supported beam

The slope deflection method (Euler beam theory) gives the system of equation as

$$\mathbf{K}\mathbf{u} = \mathbf{p},$$

where

$$\mathbf{K} = \frac{EI}{L}\begin{bmatrix} \ddots & & & & & \\ & 4+4 & 2 & & & \\ & 2 & 4+4 & 2 & & \\ & & 2 & 4+4 & 2 & \\ & & & 2 & 4+4 & \\ & & & & & \ddots \end{bmatrix}, \; \mathbf{u} = \begin{pmatrix} \vdots \\ \theta_{l-1} \\ \theta_l \\ \theta_r \\ \theta_{r+1} \\ \vdots \end{pmatrix}, \; \mathbf{p} = \begin{pmatrix} \vdots \\ 0 \\ \frac{p}{4} \\ -\frac{p}{4} \\ 0 \\ \vdots \end{pmatrix}.$$

Let $\mathbf{K}_N$ denote a truncated stiffness matrix with size $2N \times 2N$ and $\mathbf{u}_N$ and $\mathbf{p}_N$ are the truncated load and displacement vectors, i.e.,

$$\mathbf{K}_N = \frac{EI}{L}\begin{bmatrix} 8 & 2 & & & & & & \\ 2 & \ddots & & & & & & \\ & & 8 & 2 & & & & \\ & & 2 & 8 & 2 & & & \\ & & & 2 & 8 & 2 & & \\ & & & & 2 & 8 & & \\ & & & & & & \ddots & 2 \\ & & & & & & 2 & 8 \end{bmatrix}, \; \mathbf{u}_N = \begin{pmatrix} \theta_{l-N+1} \\ \vdots \\ \theta_{l-1} \\ \theta_l \\ \theta_r \\ \theta_{r+1} \\ \vdots \\ \theta_{r+N-1} \end{pmatrix}, \; \mathbf{p}_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{p}{4} \\ -\frac{p}{4} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then, the equation $\mathbf{K}_N\mathbf{u}_N = \mathbf{p}_N$ gives an approximated solution of the original problem. For example, $N = 2$, the system of equation after matrix condensation yields

$$\frac{EI}{L}\begin{bmatrix} 8 - \frac{2^2}{8} & 2 \\ 2 & 8 - \frac{2^2}{8} \end{bmatrix}\begin{pmatrix} \theta_l \\ \theta_r \end{pmatrix} = \begin{pmatrix} \frac{p}{4} \\ -\frac{p}{4} \end{pmatrix}.$$

We can generalize the above condensed equation for an arbitrary N, i.e.,

$$\frac{EI}{L}\begin{bmatrix} a_N & 2 \\ 2 & a_N \end{bmatrix}\begin{pmatrix} \theta_l \\ \theta_r \end{pmatrix} = \begin{pmatrix} \frac{p}{4} \\ -\frac{p}{4} \end{pmatrix}.$$

In the above,

$$a_{n+1} = 8 - \frac{2^2}{a_n}, \; n = 1, 2, \ldots, N-1 \text{ and } a_1 = 4.$$

The above sequence converges to $4 + 2\sqrt{3} \approx 7.4641$ as $n \to \infty$. Note that 10 iterations are sufficient to meet $\frac{\|a_{n+1}-a_n\|}{\|a_n\|} < 10^{-1}$. Finally we have the solution for the infinitely-long bridge with periodic supports:

$$\begin{pmatrix} \theta_l \\ \theta_r \end{pmatrix} = \frac{pL}{4EI}\begin{pmatrix} \frac{1}{2+2\sqrt{3}} \\ -\frac{1}{2+2\sqrt{3}} \end{pmatrix} \approx \frac{pL}{EI}\begin{pmatrix} 0.0458 \\ -0.0458 \end{pmatrix}.$$