Numerical dispersion of finite difference and finite element methods

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Abstract

We analyze the dispersion characteristics of finite element (FE) and finite difference (FD) approximations of a wave equation in the context of a discrete periodic medium.

1 Mass-spring system: a discrete periodic medium

We first consider a simple discrete periodic medium, i.e., an infinite train of mass-spring system with the periodicity of p (Figure 1). The governing equation for the system is

$$\begin{aligned} 0 &= K\left(u_{n-1}-u_n\right)-K\left(u_n-u_{n+1}\right)-M\frac{\partial^2 u_n}{\partial t^2} \\ &= K\left(u_{n-1}-2u_n+u_{n+1}\right)-M\frac{\partial^2 u_n}{\partial t^2} \quad n\in\mathbb{Z}, \end{aligned} \tag{1}$$

where u_n is a displacement at each node, M is a mass, and K is a spring constant.

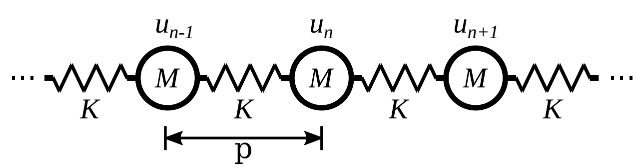

Figure 1: Schematic of an infinite train of mass-spring system

We assume a plane-wave-like ansatz, i.e.,

$$u_n = Ae^{\mathrm{i}(knp-\omega t)}. \tag{2}$$

In the above, A is a constant and ω is a circular frequency. The ansatz satisfies

$$u_{n+1} = e^{\mathrm{i}kp}u_n \quad \text{and} \tag{3a}$$

$$u_{n-1} = e^{-\mathrm{i}kp}u_n, \tag{3b}$$

which are the Floquet-Bloch boundary condition of discrete periodic systems. Plugging (2) and (3) into the governing equation (1), we have an eigenvalue problem for kd, i.e.,

$$\begin{aligned} 0 &= K\left(e^{-\mathrm{i}kp}-2+e^{\mathrm{i}kp}\right)u_n+M\omega^2 u_n \\ &= \left(-4K\sin^2\frac{kp}{2}+M\omega^2\right)u_n. \end{aligned} \tag{4}$$

Then, for a given $\omega \in \mathbb{R}$, the condition for having nontrivial solutions is

$$\frac{\omega^2}{\omega_{os}^2} = 4\sin^2\frac{kp}{2}, \tag{5}$$

where $\omega_{os} = \sqrt{K/M}$ is introduced to normalize the frequency. Figure 2 shows the dispersion relation of the mass-spring system. The dispersion curve is symmetric about $kp = 0$ and periodic with a periodicity of 2π in kp. The phase and the group velocities for $kp \in [0, \pi]$ are

$$v_{ph} = \frac{\omega}{k} = \omega_{os} p \frac{\sin(kp/2)}{kp/2} \quad \text{and} \tag{6}$$

$$v_{gr} = \frac{\partial \omega}{\partial k} = \omega_{os} p \cos\frac{kp}{2}. \tag{7}$$

Around the origin, or a small kp, we have nearly linear dispersion, where $v_{ph} = v_{gr} = \omega_{os} p$. No propagating wave exists above the cutoff frequency $\omega_c = 2\omega_{os}$.

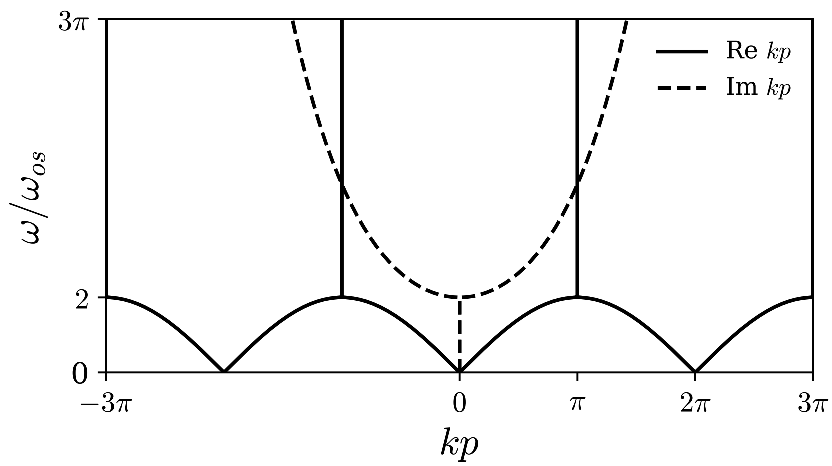

Figure 2: Dispersion relation of an infinite train of mass-spring system

2 One-dimensional wave propagation

We compare the dispersion characteristics of the analytical solution and its finite difference- and finite element-approximations of a one-dimensional wave equation.

A one-dimensional wave equation in an unbounded domain reads

$$0 = \mu \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} \quad x \in \mathbb{R}, \tag{8}$$

where ρ is a density and μ is a modulus. Then, the dispersion relation is

$$0 = \mu k^2 - \rho\, \omega^2. \tag{9}$$

Thus, we have a linear relation between k and ω, or a non-dispersive medium, where the phase and group velocities are

$$v_{ph} = \frac{\omega}{k} = \sqrt{\frac{\mu}{\rho}} \equiv c \quad \text{and} \tag{10}$$

$$v_{gr} = \frac{\partial \omega}{\partial k} = c. \tag{11}$$

2.1 Finite difference method

Here, we approximate the spatial derivatives of the wave equation using a finite difference method (FDM). When a central differencing scheme is used with the sampling interval of p, the wave equation can be written as a mass-spring system as (1), i.e.,

$$\begin{aligned} 0 =& \mu \frac{u_{n-1} - 2u_n + u_{n+1}}{p^2} - \rho \frac{\partial^2 u_n}{\partial t^2} \\ =& \frac{\mu}{p} \left(u_{n-1} - 2u_n + u_{n+1}\right) - \rho p \frac{\partial^2 u_n}{\partial t^2} \\ =& K_{\text{eff}} \left(u_{n-1} - 2u_n + u_{n+1}\right) - M_{\text{eff}} \frac{\partial^2 u_n}{\partial t^2}. \end{aligned} \tag{12}$$

In above, the effective mass and the spring constant are defined by

$$M_{\text{eff}} = \rho p \quad \text{and} \tag{13a}$$

$$K_{\text{eff}} = \frac{\mu}{p}. \tag{13b}$$

Thus, the central difference scheme yields to a discrete periodic medium, which have the same dispersive characteristics with those of the mass-spring system.

Figure 3 shows the dispersion curves of an analytical and a numerical (central differencing scheme) solutions. Similar to the mass-spring system, $\omega_{os} = \sqrt{K_{\text{eff}}/M_{\text{eff}}}$ is introduced to normalize the frequency ω. Compared to the non-dispersive analytical solution, FDM has a numerical dispersion, which is periodic in kp and under estimates the frequency ω. The periodicity results in aliasing for $|kp| > \pi$. Thus, for the wavelength of $\lambda = 2\pi/k$, the minimum sampling interval to avoid aliasing is $\lambda/2$, which is the Nyquist frequency. Moreover, the FDM cannot support a wave propagation when the frequency ω is above the cutoff frequency $\omega_c = 2\omega_{os}$. We define a relative error of the frequency as

$$\|e\|_{(a,b)} = \left[\frac{\int_a^b \left|\omega_{\text{numerical}} - \omega_{\text{analytical}}\right|^2 d\left(kp\right)}{\int_a^b \left|\omega_{\text{analytical}}\right|^2 d\left(kp\right)} \right]^{1/2}. \tag{14}$$

Then, we have $\|e\|_{(0,\pi/4)} \approx 1.67\,\%$ and $\|e\|_{(0,\pi)} \approx 24.47\,\%$, where $p = \lambda/8$ for $kp = \pi/4$ and $p = \lambda/2$ for $kp = \pi$.

2.2 Finite element method

The wave equation approximated by a finite element method is

$$0 = -u_n \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \frac{\partial \varphi_m}{\partial x} \mu \frac{\partial \varphi_n}{\partial x} dx - \frac{\partial^2 u_n}{\partial t^2} \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \varphi_m \rho \varphi_n dx \quad n \in \mathbb{Z}. \tag{15}$$

Here, we use linear shape functions, i.e.,

$$\varphi_n = \begin{cases} \left(x + p - pn\right)/p & x \in \left(pn - p, pn\right) \\ \left(p + pn - x\right)/p & x \in \left(pn, pn + p\right) \\ 0 & \text{otherwise} \end{cases}. \tag{16}$$

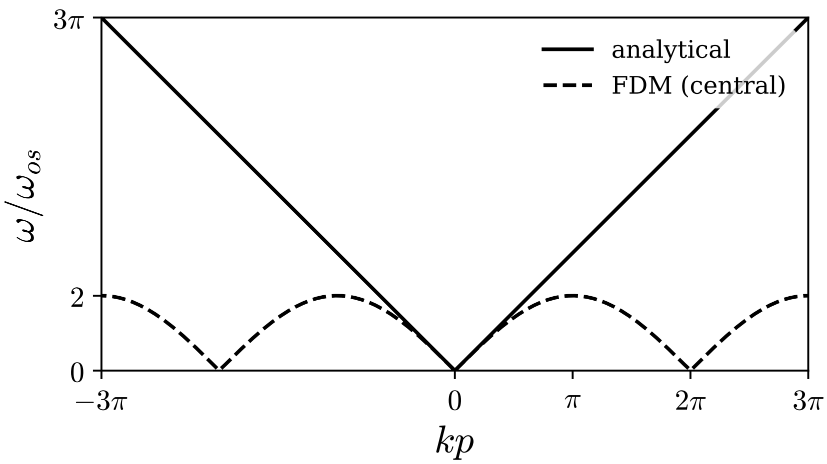

Figure 3: Dispersion curves of analytical, and FD solutions

Then, the above equation (15) can be written as

$$\begin{aligned} 0 &= \frac{\mu}{p}\left(u_{n-1} - 2u_n + u_{n+1}\right) - \frac{\rho p}{3}\left(\frac{1}{2}\frac{\partial^2 u_{n-1}}{\partial t^2} + 2\frac{\partial^2 u_n}{\partial t^2} + \frac{1}{2}\frac{\partial^2 u_{n+1}}{\partial t^2}\right) \\ &= \frac{\mu}{p}\left(u_{n-1} - 2u_n + u_{n+1}\right) - \rho p\left(1 - \frac{2}{3}\sin^2\frac{kp}{2}\right)\frac{\partial^2 u_n}{\partial t^2} \\ &= K_{\text{eff}}\left(u_{n-1} - 2u_n + u_{n+1}\right) - M_{\text{eff}}\frac{\partial^2 u_n}{\partial t^2}, \end{aligned} \tag{17}$$

where

$$M_{\text{eff}} = \rho p\left(1 - \frac{2}{3}\sin^2\frac{kp}{2}\right) \quad \text{and} \tag{18a}$$

$$K_{\text{eff}} = \frac{\mu}{p}. \tag{18b}$$

The dispersion relation is given as

$$\frac{\omega^2}{\omega_{os}^2} = \frac{4\sin^2\frac{kp}{2}}{1 - \frac{2}{3}\sin^2\frac{kp}{2}}, \tag{19}$$

where

$$\omega_{os} = \lim_{kp\to 0}\sqrt{\frac{K_{\text{eff}}}{M_{\text{eff}}}}. \tag{20}$$

Thus, the FE approximation can also be regarded as a discrete periodic medium with frequency-dependent effective mass. Figure 4 shows the dispersion curve of the FE approximation. Similar to the FDM case, a cutoff frequency exists at $\omega_c = 2\sqrt{3}\omega_{os}$ and the dispersive curve is symmetric and periodic in kp with periodicity of 2π, thus, aliasing is present. The dispersion curve of FEM overestimates the frequency compared to the analytical solution, however, the curve is more "linear" compared to that of FDM. the relative error of the frequency ω is summarized in table 1.

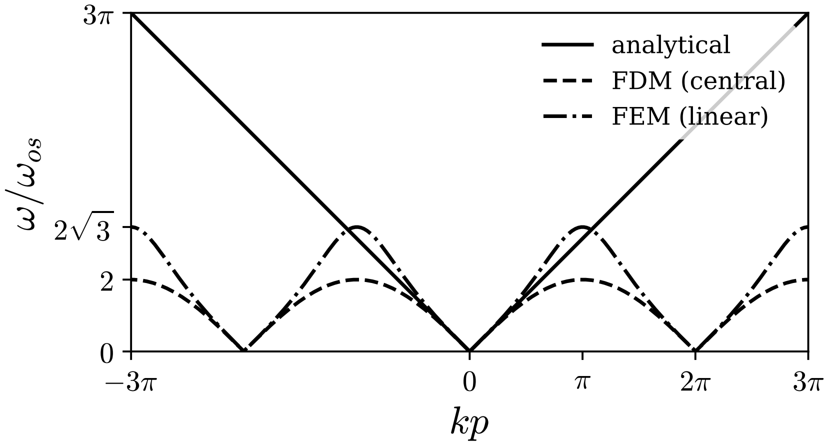

Figure 4: Dispersion curves of analytical, FD, and FE approximations

Table 1: Relative error $\|e\|_{(a,b)}$

(a,b)	FDM (central)	FEM (linear)
$(0,\pi/4)$	1.67 %	1.69 %
$(0,\pi)$	24.47 %	16.12 %