Gauss Quadrature

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1 Gauss-Legendre quadrature

Gauss-Legendre quadrature is exact for polynomial functions upto $2N+1$-th order, where N is the number of Gauss points.

1D line An integral of a function $f(\xi)$ can be approximated by

$$\int_{1}^{-1} f(\xi) d\xi \approx \sum_{i=1}^{N} f(\xi_i) w_i. \tag{1}$$

In the above ξ_i are the Gauss points and w_i are weights.

Gauss points and weights for $[-1, 1]$ are listed below:

- $N = 1$

$$\xi_1 = 0, \quad w_1 = 2$$

- $N = 2$

$$\xi_1 = -\frac{1}{\sqrt{3}}, \quad w_1 = 1$$
$$\xi_2 = \frac{1}{\sqrt{3}}, \quad w_2 = 1$$

- $N = 3$

$$\xi_1 = -\sqrt{\frac{3}{5}}, \quad w_1 = \frac{5}{9}$$
$$\xi_2 = 0, \quad w_2 = \frac{8}{9}$$
$$\xi_3 = \sqrt{\frac{3}{5}}, \quad w_3 = \frac{5}{9}$$

- See `https://en.wikipedia.org/wiki/Gaussian_quadrature` for higher orders.

For a general interval $[a, b]$, use the transformation below:

$$x = \frac{b-a}{2}\xi + \frac{a+b}{2}. \tag{2}$$

Then, with $dx/d\xi = (b-a)/2$, we have

$$\int_{b}^{a} f(x) dx \approx \frac{b-a}{2} \sum_{i=1}^{N} f\left(\frac{b-a}{2}\xi_i + \frac{a+b}{2}\right) w_i. \tag{3}$$

3D quadrilateral For $[-1,1]\times[-1,1]\times[-1,1]$, we have

$$\int_{1}^{-1}\int_{1}^{-1}\int_{1}^{-1} f(\xi,\eta,\zeta)d\xi d\eta d\zeta \approx \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} f(\xi_i,\eta_j,\zeta_k)w_i w_j w_k. \tag{4}$$

For a general interval $[x_a,x_b]\times[y_a,y_b]\times[z_a,z_b]$, use

$$\int_{z_a}^{z_b}\int_{y_a}^{y_b}\int_{x_a}^{x_b} f(x,y,z)dxdydz \approx \frac{x_b-x_a}{2}\frac{y_b-y_a}{2}\frac{z_b-z_a}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} f(x_i,y_j,z_k)w_i w_j w_k, \tag{5}$$

where

$$x_i = \frac{x_b-x_a}{2}\xi_i + \frac{x_a+x_b}{2}, \quad y_j = \frac{y_b-y_a}{2}\eta_j + \frac{y_a+y_b}{2}, \quad z_k = \frac{z_b-z_a}{2}\zeta_k + \frac{z_a+z_b}{2}. \tag{6}$$