Approximation of functions

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1 Change of basis

The (contravariant) component of a vector v^i corresponds to an orthonomal basis $\mathbf{e}_i$ can be easily obtained by computing its projection on the basis, i.e.,

$$(\mathbf{v}, \mathbf{e}_i) = v^i. \tag{1}$$

However, we need a rigorous approach for non-orthonormal bases; the contravariant component of a vector v^i is obtained by its projection on the dual basis $\mathbf{g}^i$, i.e.,

$$\left(\mathbf{v}, \mathbf{g}^i\right) = \left(v^j \mathbf{g}_j, \mathbf{g}^i\right) = v^j \left(\mathbf{g}_j, \mathbf{g}^i\right) = v^j \delta^i_j = v^i. \tag{2}$$

Similary, we can calculate the covariant component of a vector by

$$(\mathbf{v}, \mathbf{g}_i) = v_i. \tag{3}$$

We can still calculate the contravariant components of a vector with basis, however, in a sligthly convoluted way, i.e.,

$$\begin{aligned} (\mathbf{v}, \mathbf{g}_i) &= \left(v^j \mathbf{g}_j, \mathbf{g}_i\right) \\ &= v^j \left(\mathbf{g}_j, \mathbf{g}_i\right). \end{aligned} \tag{4}$$

In the above, we have a system of equations, where v^j are the unknowns. Thus, in general, we need to solve for all v^j, simultaneously. The above system of equations can be decoupled when the basis is orthogonal, i.e.,

$$v^i = \frac{(\mathbf{v}, \mathbf{g}_i)}{(\mathbf{g}_i, \mathbf{g}_i)}. \tag{5}$$

For an orthonormal basis, (5) reduces to (1).

Example 1.1 (Coordinate transform) *Calculate the component of a vector* $\mathbf{v} = 2\mathbf{e}_1 + \mathbf{e}_2$ *with respect to basis* $\mathbf{g}_1 = \mathbf{e}_1$ *and* $\mathbf{g}_2 = \mathbf{e}_1 + \mathbf{e}_2$.
Expanding the (4), we have

$$v^1 \mathbf{g}_1 \cdot \mathbf{g}_1 + v^2 \mathbf{g}_2 \cdot \mathbf{g}_1 = \mathbf{v} \cdot \mathbf{g}_1 \quad \text{and} \tag{6a}$$

$$v^1 \mathbf{g}_1 \cdot \mathbf{g}_2 + v^2 \mathbf{g}_2 \cdot \mathbf{g}_2 = \mathbf{v} \cdot \mathbf{g}_2. \tag{6b}$$

In the above, we have $\mathbf{g}_1 \cdot \mathbf{g}_1 = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1$, $\mathbf{g}_1 \cdot \mathbf{g}_2 = \mathbf{g}_2 \cdot \mathbf{g}_1 = \mathbf{e}_1 \cdot (\mathbf{e}_1 + \mathbf{e}_2) = 1$, $\mathbf{g}_2 \cdot \mathbf{g}_2 = (\mathbf{e}_1 + \mathbf{e}_2) \cdot (\mathbf{e}_1 + \mathbf{e}_2) = 2$, $\mathbf{v} \cdot \mathbf{g}_1 = (2\mathbf{e}_1 + \mathbf{e}_2) \cdot \mathbf{e}_1 = 2$, and $\mathbf{v} \cdot \mathbf{g}_2 = (2\mathbf{e}_1 + \mathbf{e}_2) \cdot (\mathbf{e}_1 + \mathbf{e}_2) = 3$. Rewriting (6) into a matrix equation, we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \tag{7}$$

Then, the solution to the above equation is

$$v^1 = 1, \quad \text{and} \quad v^2 = 1. \tag{8}$$

Alternatively, we can obtain the same result using dual bases, i.e., $\mathbf{g}^1 = \mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{g}^2 = \mathbf{e}_2$. Then, the equation (2) gives

$$v^1 = \mathbf{v} \cdot \mathbf{g}^1 = (2\mathbf{e}_1 + \mathbf{e}_2) \cdot (\mathbf{e}_1 - \mathbf{e}_2) = 1 \quad \text{and} \tag{9a}$$

$$v^2 = \mathbf{v} \cdot \mathbf{g}^2 = (2\mathbf{e}_1 + \mathbf{e}_2) \cdot \mathbf{e}_2 = 1. \tag{9b}$$

In summary, we have $\mathbf{v} = 2\mathbf{e}_1 + \mathbf{e}_2 = \mathbf{g}_1 + \mathbf{g}_2$.

Example 1.2 (Fourier series) *Consider an odd function $f(x)$ defined as below*

$$f(x) = x, \quad x \in (-0.5, 0.5). \tag{10}$$

Find the component of sinusoidal basis, i.e.,

$$g_n = \sin(n\pi x), \quad n \in \mathbb{Z}_{++}. \tag{11}$$

The above sinusoidal basis is orthogonal; thus, we use (5), where

$$(g_n, g_n) = \int_{-0.5}^{0.5} \sin(n\pi x) \sin(n\pi x)\, dx = 0.5, \quad \forall n \in \mathbb{Z}_{++} \quad \text{and} \tag{12a}$$

$$(f, g_n) = \int_{-0.5}^{0.5} x \sin(n\pi x)\, dx = \frac{2\sin(n\pi/2)}{(n\pi)^2} - \frac{\cos(n\pi/2)}{n\pi}. \tag{12b}$$

Then, we have

$$v^n = \frac{(f, g_n)}{(g_n, g_n)} = \frac{\sin(n\pi/2)}{(n\pi)^2} - \frac{\cos(n\pi/2)}{2n\pi}. \tag{13}$$

Finally, the function $f(x)$ is *discretized*, or expressed by linear combinations of basis g_n, i.e.,

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{\sin(n\pi/2)}{(n\pi)^2} - \frac{\cos(n\pi/2)}{2n\pi} \right) \sin(n\pi x). \tag{14}$$

2 Approximation

What will happen if the choice of basis does not span the function space, e.g., a truncated Fourier series? In such case, we have an approximation of a function. In fact, we obtain the best approximation with respect to the norm induced by the associated inner product. We can verify this by formulating an optimization problem, or least square error problem: given f and g_n, find v^n such that

$$\min \Pi, \quad \Pi = \frac{1}{2} \left\| v^n g_n - f \right\|^2. \tag{15}$$

In the above, the objective functional Π is proportional to the square of the error, measured by the induced norm, i.e.,

$$\|a\| = \sqrt{(a, a)}. \tag{16}$$

The objective functional Π vanishies if and only if $v^n g_n = f$; otherwise, it will always return a positive non-zero number. Then, the minimization problem can be solved by satisfying the first optimality condition, i.e.,

$$\begin{aligned} 0 &= \frac{\partial \Pi}{\partial v^n} \\ &= \frac{1}{2}\frac{\partial}{\partial v^n}\left(v^m g_m - f, v^m g_m - f\right) \\ &= \left(v^m g_m - f, g_n\right) \\ &= v^m \left(g_m, g_n\right) - \left(g_n, f\right). \end{aligned} \tag{17}$$

Thus, we have recovered the same expression as (4).

Example 2.1 (Polynomial approximation) *Approximate a function* $f(x) = \sin x$*,* $x \in (-\pi, \pi)$ *using polynomials* $g_n = x^n$*,* $n = 1, 2, \ldots, N$*,* $N = 5$*.*
We use (4), where

$$(g_m, g_n) = \int_{-\pi}^{\pi} x^{m+n} dx = \frac{\pi^{m+n+1} - (-\pi)^{m+n+1}}{m+n+1}, \tag{18a}$$

$$(g_1, \sin x) = 2\pi, \tag{18b}$$

$$(g_2, \sin x) = 0, \tag{18c}$$

$$(g_3, \sin x) = 2\pi\left(\pi^2 - 6\right), \tag{18d}$$

$$(g_4, \sin x) = 0, \quad \text{and} \tag{18e}$$

$$(g_5, \sin x) = 2\pi\left(120 - 20\pi^2 + \pi^4\right). \tag{18f}$$

Then, we have $v^1 \approx 0.9879$, $v^2 = 0$, $v^3 \approx -0.1553$, $v^4 = 0$, and $v^5 \approx 0.0056$. Figure 1 shows the polynomial approximations of function f using $N = 3$ and $N = 5$. ♣

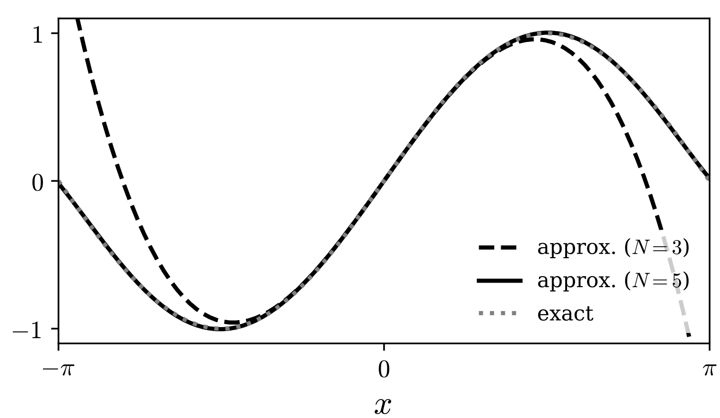

Figure 1: Polynomial approximations of $\sin x$ (Example 2.1)

Different choice of bases yield different accuracy, computational costs, stability, etc. Finite element method is one example, where we discretize a differential equation, therefore a function/solution, by a set of "local" basis.