Final Exam - Structural Analysis 1

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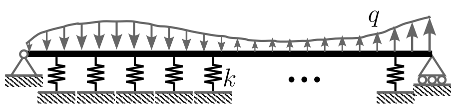
December 15th, 2025

Problem 1. A simply supported beam resting on an elastic foundation may be modeled by a continuous distribution of linear springs with stiffness k. The governing equation reads

$$\begin{cases} \dfrac{d^2}{dx^2}\left[EI\dfrac{d^2w}{dx^2}\right] + kw = q, & x \in (0, L) \\ w = \dfrac{d^2w}{dx^2} = 0, & x = 0 \\ w = \dfrac{d^2w}{dx^2} = 0, & x = L \end{cases}. \tag{1}$$

Derive the corresponding principle of virtual work.



Problem 1.

Problem 2. Consider a beam with fixed boundaries under two loading cases. Each problem is defined respectively as

$$\begin{cases} EI\dfrac{d^4w_1}{dx^4} = q_1, & x \in (0, L) \\ w_1 = \dfrac{dw_1}{dx} = 0, & x = 0 \\ w_1 = \dfrac{dw_1}{dx} = 0, & x = L \end{cases} \quad \text{and} \quad \begin{cases} EI\dfrac{d^4w_2}{dx^4} = q_2, & x \in (0, L) \\ w_2 = \dfrac{dw_2}{dx} = 0, & x = 0 \\ w_2 = \dfrac{dw_2}{dx} = 0, & x = L \end{cases}. \tag{2}$$

The loadings are specified as

$$q_1(x) = \delta(x - \xi), \ \xi \in (0, L) \quad \text{and} \quad q_2(x) = \delta''\left(x - \frac{L}{2}\right). \tag{3}$$

Here, δ and δ'' are Dirac delta and its second-order derivative, respectively. Compare the implication of the reciprocity relation

$$\int_0^L q_1 w_2 dx - [V_1 w_2]_{x=0}^{x=L} + [M_1\theta_2]_{x=0}^{x=L} = \int_0^L q_2 w_1 dx - [V_2 w_1]_0^{x=L} + [M_2\theta_1]_{x=0}^{x=L} \tag{4}$$

with the Müller-Breslau's principle to obtain the influence line for an internal bending moment.

Problem 3. The principle of minimum potential energy implies that the solution to a simple beam problem is the minimizer of the following problem:

$$\min M, \quad M[w] = \frac{1}{2}\int_0^L \frac{d^2w}{dx^2} EI \frac{d^2w}{dx^2} dx - \int_0^L wq dx. \tag{5}$$

Here, q is an external load and w is an admissible function satisfying boundary conditions: $w(0) = w(L) = 0$ and $EI\frac{d^2}{dx^2}w(0) = EI\frac{d^2}{dx^2}w(L) = 0$.

Now, consider a case when there is a support at $x = L/2$ with a settlement $\Delta \in \mathbb{R}$ downward. Then, the above minimization problem can be augmented by a Lagrange multiplier, where the Lagrangian L reads

$$L[\lambda, w] = M[w] - E[\lambda, w], \quad E[\lambda, w] = \lambda \int_0^L \delta\left(x - \frac{L}{2}\right)[w + \Delta] dx. \tag{6}$$

Here, $\lambda \in \mathbb{R}$ is called a Lagrange multiplier and $\delta\left(x - \frac{L}{2}\right)$ is Dirac delta. The necessary conditions for the solution to the above problem are

$$\begin{aligned} 0 &= d_\lambda L[\lambda, w]\left(\tilde{\lambda}\right) \\ &= -\tilde{\lambda}\int_0^L \delta\left(x - \frac{L}{2}\right)[w + \Delta] dx, \quad \forall\tilde{\lambda} \quad \text{and} \end{aligned} \tag{7}$$

$$\begin{aligned} 0 &= d_w L[\lambda, w](\tilde{w}) \\ &= \int_0^L \frac{d^2\tilde{w}}{dx^2} EI \frac{d^2w}{dx^2} dx - \int_0^L \tilde{w} q dx - \lambda \int_0^L \delta\left(x - \frac{L}{2}\right)\tilde{w} dx, \quad \forall\tilde{w}. \end{aligned} \tag{8}$$

In the above, we have two equations, (7) and (8), and two unknowns: w and λ. $\tilde{w}$ and $\tilde{\lambda}$ are the arbitrary directions or test functions.

(a) The strong form of the condition (7) is obtained by simply removing the dependency of $\tilde{\lambda}$, i.e.,

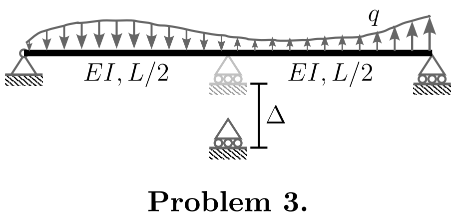
$$w\left(\frac{L}{2}\right) + \Delta = 0. \tag{9}$$

Perform integration by parts on the condition (8) to obtain the strong form:

$$\begin{cases} \dfrac{d^2}{dx^2}\left[EI\dfrac{d^2w}{dx^2}\right] = q + \lambda\delta\left(x - \dfrac{L}{2}\right), \quad x \in (0, L) \\ w(0) = w(L) = 0 \\ EI\dfrac{d^2}{dx^2}w(0) = EI\dfrac{d^2}{dx^2}w(L) = 0 \end{cases}. \tag{10}$$

(b) Thus, the original problem reduces to solving (10) subject to (9). Interpret the conditions (9) and (10) in the framework of analyzing statically indeterminate structures via linear superposition of primary structures. Explain the physical meaning of λ.

(c) Determine λ for the case $q = 0$.



Problem 3.

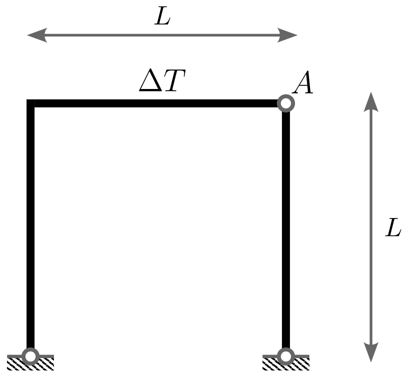
Problem 4. Consider a three-hinged frame subjected to a linear thermal gradient applied to the top horizontal member.

(a) Sketch the resulting deformation shape.

(b) Determine the horizontal displacement of node A.

Recall that a linear thermal gradient induces a uniform curvature, i.e., $\frac{d^2w}{dx^2} = \frac{\alpha\Delta T}{h}$. For a beam of length L with no rotational restraint at either end, the deflection is given by

$$w = \frac{\alpha\Delta T}{2h}x^2 - \frac{\alpha\Delta T L}{2h}x. \tag{11}$$



Problem 4.

Integration table.

M_A \ M_B	M_1; L	M_1, M_2; L	M_1; a, b; L	M_1; L
M_3; L	$\frac{L}{2}M_1M_3$	$\frac{L}{2}(M_1+M_2)M_3$	$\frac{L}{2}M_1M_3$	$\frac{2L}{3}M_1M_3$
M_3; L	$\frac{L}{3}M_1M_3$	$\frac{L}{6}(M_1+2M_2)M_3$	$\frac{L}{6}\left(1+\frac{a}{L}\right)M_1M_3$	$\frac{L}{3}M_1M_3$
M_3; L	$\frac{L}{6}M_1M_3$	$\frac{L}{6}(2M_1+M_2)M_3$	$\frac{L}{6}\left(1+\frac{b}{L}\right)M_1M_3$	$\frac{L}{3}M_1M_3$
M_3, M_4; L	$\frac{L}{6}M_1(M_3+2M_4)$	$\frac{L}{6}M_1(2M_3+M_4)$ $+\frac{L}{6}M_2(M_3+2M_4)$	$\frac{L}{6}\left(1+\frac{b}{L}\right)M_1M_3$ $+\frac{L}{6}\left(1+\frac{a}{L}\right)M_1M_4$	$\frac{L}{3}M_1(M_3+M_4)$
M_3; c, d; L	$\frac{L}{6}\left(1+\frac{c}{L}\right)M_1M_3$	$\frac{L}{6}\left(1+\frac{d}{L}\right)M_1M_3$ $+\frac{L}{6}\left(1+\frac{c}{L}\right)M_2M_3$	for $c \le a$, $\frac{L}{3}M_1M_3$ $-\frac{L(a-c)^2}{6ad}M_1M_3$	$\frac{L}{3}\left(1+\frac{cd}{L^2}\right)M_1M_3$
M_3; L	$\frac{L}{3}M_1M_3$	$\frac{L}{3}(M_1+M_2)M_3$	$\frac{L}{3}\left(1+\frac{ab}{L^2}\right)M_1M_3$	$\frac{8L}{15}M_1M_3$
M_3; L	$\frac{L}{4}M_1M_3$	$\frac{L}{12}(M_1+3M_2)M_3$	$\frac{L}{12}\left(1+\frac{a}{L}+\frac{a^2}{L^2}\right)M_1M_3$	$\frac{L}{5}M_1M_3$