Heedong Goh

Lecture Notes for

Mechanics of Materials

Department of Civil and Environmental Engineering
Seoul National University
2025

Preface

This monograph is written for the course *Mechanics of Materials* offered in the Department of Civil and Environmental Engineering at Seoul National University.

In preparing these notes, I borrowed many parts from the following books and lecture notes: [Crandall et al., 2012, Hibbeler, 2010, Goodno and Gere, 2017, Lee, 2022a, Lee, 2022b].

These lecture notes are a working manuscript, subject to ongoing refinement and enhancement; I welcome and greatly appreciate any reports regarding errors, typos, or any other inaccuracies found within the notes.

Contents

Chapter 1

Preliminaries

1.1 Numbers, vectors, and tensors

A *field* is a *set* equipped with two operations: addition and multiplication satisfying associativity and distributivity. Each operation is a *group*, where every member has an additive inverse and every nonzero member has an multiplicative inverse. For example, the real numbers form a field denoted by $\mathbb{R}$ and the complex numbers form a field denoted by $\mathbb{C}$.

Vectors are objects that can be added to each other and scaled by numbers to produce new vectors. We say that two vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$, are *linearly independent* when

$$\alpha \boldsymbol{u} + \beta \boldsymbol{v} = \boldsymbol{0} \tag{1.1}$$

is satified only if the numbers are $\alpha = \beta = 0$. A vector space has *dimension* n if the space has n linearly independent vectors but any set of $n + 1$ vectors is linearly dependent. Then, any set of n lineary independent vectors can be choosen as a *basis*, where an arbitrary vector can be expressed by

$$\boldsymbol{v} = \sum_{i=1}^{N} v^i \boldsymbol{g}_i = v^1 \boldsymbol{g}_1 + v^2 \boldsymbol{g}_2 + \ldots + v^N \boldsymbol{g}_N. \tag{1.2}$$

In the above, $\boldsymbol{g}_i$ are the basis vectors and v^i are the components. Throughout these notes, we will always use *orthonormal* basis, i.e., *Cartesian*, where for all basis vectors $\boldsymbol{g}_i$, we have

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_j = 0 \quad \forall i \neq j \quad \text{and} \tag{1.3}$$

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_i = 1. \tag{1.4}$$

Here, $\cdot$ is a single contraction, i.e., $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{N} a^i b_i$. Formally, it is defined using the concept of *dual vectors*, where their components is denoted by a lower index. However, vectors and dual vectors become identical if an orthonormal basis is used; thus, we will use lower indices for denoting vectors and their dual from now on.

Tensors are generalization of numbers composed of vectors by a tensor operation $\otimes$, which is defined as

$$(\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\, \boldsymbol{a}. \tag{1.5}$$

Thus, a tensor is identified as a linear map of vectors to vectors. For example, a second-order reads

$$\begin{aligned}\mathbf{A} = \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}\mathbf{g}_i \otimes \mathbf{g}_j = A_{11}\mathbf{g}_1 \otimes \mathbf{g}_1 + A_{12}\mathbf{g}_1 \otimes \mathbf{g}_2 + \dots \\ + A_{NN}\mathbf{g}_N \otimes \mathbf{g}_N,\end{aligned} \tag{1.6}$$

where its operation on a vector $\boldsymbol{v} = \sum_{i=1}^{N} v_i \boldsymbol{g}_i$ is

$$\begin{aligned}Av &= \left(\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\right)\sum_{k=1}^{N} v_k \boldsymbol{g}_k \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} A_{ij} v_k \left(\boldsymbol{g}_i \otimes \boldsymbol{g}_j\right)\boldsymbol{g}_k \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} A^{ij} v_k \left(\boldsymbol{g}_j \cdot \boldsymbol{g}_k\right)\boldsymbol{g}_i \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij} v_j \boldsymbol{g}_i.\end{aligned} \tag{1.7}$$

Here, we recovered the usual matrix-vector operation. The order of tensors can be higher than two; for example a third-order tensor reads

$$\begin{aligned}B = &\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} B_{ijk}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k \\ = &B_{111}\boldsymbol{g}_1 \otimes \boldsymbol{g}_1 \otimes \boldsymbol{g}_1 + B_{112}\boldsymbol{g}_1 \otimes \boldsymbol{g}_1 \otimes \boldsymbol{g}_2 + \dots \\ &+ B_{NNN}\boldsymbol{g}_N \otimes \boldsymbol{g}_N \otimes \boldsymbol{g}_N.\end{aligned} \tag{1.8}$$

In this context, a number can be considered as a tensor of zero order.

Additionally, we will omit summation symbols when expanding the components of vectors and tensors. For example,

$$\boldsymbol{v} = \sum_{i=1}^{N} v_i \boldsymbol{g}_i = v_i \boldsymbol{g}_i \quad \text{and} \quad \boldsymbol{A} = \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = A_{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j. \tag{1.9}$$

We further omit the basis vector because we will always use the Cartesian. For instance, the operation of a matrix on a vector is expressed as

$$\boldsymbol{Av} = \sum_{j=1}^{N} A_{ij} v_j = A_{ij} v_j. \tag{1.10}$$

1.2 Algebraic and differential equations

An engineering problem often takes a form of equation solving, which is a deductive process of unraveling unknowns out of an implicit information hidden in equations. The operators of algebraic equations are $+$, $-$, $\times$, and $\div$. For

example, a system of linear algebraic equations reads:
Given $A \in \mathbb{R}^{M \times N}$ and $b \in \mathbb{R}^M$, find $x \in \mathbb{R}^N$ such that $Ax = b$ or

$$\left\{ \begin{array}{c} A_{11}x_1 + A_{12}x_2 + \ldots + A_{1N}x_N = b_1 \\ A_{21}x_1 + A_{22}x_2 + \ldots + A_{2N}x_N = b_2 \\ \vdots \\ A_{M1}x_1 + A_{M2}x_2 + \ldots + A_{MN}x_N = b_M \end{array} \right. \tag{1.11}$$

or

$$\sum_{j=1}^{N} A_{ij}x_j = b_i, \; i = 1, 2, \ldots, M. \tag{1.12}$$

The unknowns of algebraic equations x_j, are *numbers*, where the number of unknowns, or *dimension*, N is typically finite. We also have a finite number of equations denoted by M. Thanks to the linearity, we can find out whether the problem is solvable by counting the number of independent equations, i.e.,

$$\left\{ \begin{array}{ll} N > M & \text{no unique solution (underdetermined system)} \\ N = M & \text{unique solution} \\ N < M & \text{no solution (overdetermined system)} \end{array} \right. .$$

Consider a linearly suspended mass-spring system. The equilibrium equation for each mass m_i is

$$k_i (x_i - x_{i-1}) - k_{i+1} (x_{i+1} - x_i) - m_i g = 0, \quad i = 0, 1, 2, \ldots, N. \tag{1.13}$$

Let $x_0 = 0$; we have a system of linear algebraic equations,

$$\begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \ldots & 0 \\ -k_1 & k_2 + k_3 & -k_3 & \ldots & 0 \\ 0 & -k_2 & k_2 + k_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & k_N \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} m_1 g \\ m_2 g \\ m_3 g \\ \vdots \\ m_N g \end{pmatrix}. \tag{1.14}$$

On the other hand, we seek for *functions* when solving differential equations, which involves with differential operators. For example:
Given $L : \mathcal{V} \to \mathcal{W}$ and $f \in \mathcal{W}$, find $u \in \mathcal{V}$ such that

$$\left\{ \begin{array}{lll} Lu(x) \equiv \dfrac{d^2 u(x)}{dx^2} = f(x), & x \in (0, 1) & \text{(governing equation)} \\ u(0) = u(1) = 0 & & \text{(boundary conditions)} \end{array} \right. . \tag{1.15}$$

In the above, the unknown function $u : \mathbb{R} \to \mathbb{R}$ and the corresponding governing equation are defined over an interval $x \in (0, 1)$, which excludes boundaries. The function values at the boundaries are explicitly given as boundary conditions. Unlike the algebraic equations, the dimension of a function space, such as $\mathcal{W}$ and $\mathcal{V}$, is infinite. When defining a function space, we consider a set of *admissible functions* to satisfy physical/mathematical constraints such as continuity and differentiability. Rigorous studies on function space or the existence and the uniqueness of differential equations are out of our scope.

We require n number of boundary conditions for a n-th order differential equation. For each boundary, we can specify upto $n-1$-th derivatives, where the higher half, u, du/dx, $\ldots$, $d^{n/2}u/dx^{n/2}$, are called the *Dirichlet boundary conditions* and the lower half are called the *Neumann boundary conditions.* They are also called *essential* and *natural boundary conditions*, respectively.

Consider the Newton's second law, $F = m\ddot{u}$. The general solution u can be obtained by integrating the equation twice, which gives

$$u = \frac{1}{2}\frac{F}{m}t^2 + C_1 t + C_2. \tag{1.16}$$

In the above, we are left with two undetermined constants, C_1 and C_2. The boundary, or the initial, conditions completes the analysis, e.g., $u(0) = u_0$ and $\dot{u}(0) = v_0$ gives $C_1 = v_o$ and $C_2 = u_0$. Here, we specified both Dirichlet and Neumann boundary conditions at $t = 0$, which are called *Cauchy boundary conditions.*

1.3 Miscellanies

As mentioned earlier, a *dot product*, or a single contraction, is defined as (in three dimension)

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{1.17}$$

We take the right-handed rule for the *cross product* between two vectors, i.e.,

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \det \begin{bmatrix} \boldsymbol{g}_1 & \boldsymbol{g}_2 & \boldsymbol{g}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \\ &= (a_2 b_3 - a_3 b_2)\,\boldsymbol{g}_1 + (a_3 b_1 - a_1 b_3)\,\boldsymbol{g}_2 + (a_1 b_2 - a_2 b_1)\,\boldsymbol{g}_3. \end{aligned} \tag{1.18}$$

Differentiation is defined as

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f\,(x + \Delta x) - f\,(x)}{\Delta x}. \tag{1.19}$$

Integration by parts states:

$$\int_a^b u \frac{dv}{dx} dx = [uv]_{x=a}^{x=b} - \int_a^b \frac{du}{dx} v dx, \tag{1.20}$$

where $[A(x)]_{x=a}^{x=b} = A(x)|_{x=b} - A(x)|_{x=a} = A(b) - A(a)$.

Dirac delta function is a *generalized function* or a *distribution*, acting on a regular function $f(x)$, it is defined such that [Gel'Fand and Shilov, 1964]

$$(f(x), \delta(x - x_o)) = \int_{-\infty}^{\infty} f(x)\delta(x - x_o) dx = f(x_o). \tag{1.21}$$

The above relation is called *sifting property.* The Dirac delta function can also be loosely considered as a "function" with the following properties:

$$\delta\,(x - x_o) = \begin{cases} \infty, & x = x_o \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta\,(x - x_o)\,dx = 1. \tag{1.22}$$

In the above, ∞ is not a unique value; thus, $\delta(x - x_o)$ is not a function in the conventional sense. Derivatives of Dirac delta function have following operations:

$$
\begin{aligned}
(f(x), \delta'(x - x_o)) &= \int_{-\infty}^{\infty} f(x)\delta'(x - x_o)dx \\
&= f(x)\delta(x - x_o)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x - x_o)dx \\
&= -f'(x_o), \qquad (1.23) \\
(f(x), \delta''(x - x_o)) &= f''(x_o), \qquad (1.24) \\
&\vdots \\
\Big(f(x), \delta^{(n)}(x - x_o)\Big) &= (-1)^{(n)} f^{(n)}(x_o), \quad n = 0, 1, 2, \ldots . \qquad (1.25)
\end{aligned}
$$

Chapter 2

Introduction to Mechanics of Materials

2.1 Discrete and continuum mechanics

Mechanics is a branch of physics that describes how a physical *body* displaces or deforms under an external influence, or *force*. A body consists of *materials* that exhibit *homogeneous* properties across various domains, including mechanical (our primary focus), thermal, electromagnetic, and chemical properties. However, homogeneity is a scale-dependent concept.

For instance, concrete is considered homogeneous in terms of strength when used as a construction material. Yet, at a finer scale, it comprises cement and gravel, which themselves are heterogeneous at the atomic level (Figure 2.1).

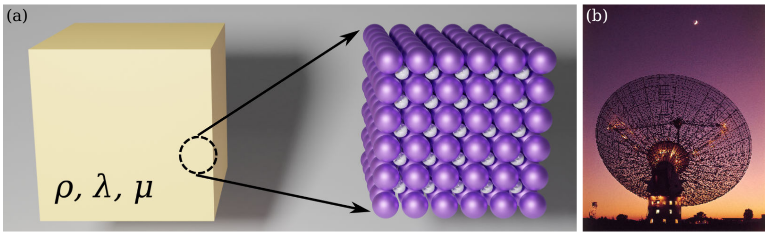


Figure 2.1: Homogeneity of materials. (a) Continuum and atomic structure. (b) CSIRO's Parkes radio telescope (`https://en.wikipedia.org/wiki/Radio_telescope`).

Readers may already be familiar with *discrete mechanics*, particularly lattice models composed of networks of masses, springs, and other discrete elements. In these lecture notes, we consider a deformable body whose material points occupy a continuous domain, i.e., a *continuum*, with mechanical properties distributed throughout the body. Physical quantities such as displacement, strain, and stress vary spatially and are expressed as functions of position. Consequently, physical phenomena are governed by differential equations rather than algebraic equations.

As an example, Figure 2.2 compares a spring system with a bar problem.

The spring problem can be formulated as follows:

Given a spring constant $k \in \mathbb{R}$ and an external force $F \in \mathbb{R}$, find the displacement $x \in \mathbb{R}$ that satisfies the equilibrium equation:

$$kx - F = 0. \tag{2.1}$$

The solution to this algebraic equation is given by:

$$x = \frac{F}{k}. \tag{2.2}$$

On the other hand, the bar problem is formulated as follows:

Given the axial rigidity $EA\left(x\right) : \mathbb{R} \to \mathbb{R}$, an external load $f\left(x\right) : \mathbb{R} \to \mathbb{R}$, and a force $F \in \mathbb{R}$ applied on the boundary, find the displacement $u\left(x\right) : \mathbb{R} \to \mathbb{R}$ that satisfies the governing equation and boundary conditions:

$$\frac{d}{dx}\left[EA\frac{du}{dx}\right] + f = 0, \quad x \in (0,1), \tag{2.3}$$

$$u(0) = 0, \tag{2.4}$$

$$EA\frac{du}{dx}\bigg|_{x=1} = F. \tag{2.5}$$

Here, f represents a distributed force applied within the domain $(0,1)$, while F is an external force prescribed on the boundary. The axial rigidity EA is the product of the *Young's modulus* E and the cross-sectional area A.

The solution is obtained by integrating the governing equation twice and determining the two integration constants using the boundary conditions. If $f = 0$ and $EA = \text{const.}$, integrating twice yields:

$$EAu = C_0 x + C_1. \tag{2.6}$$

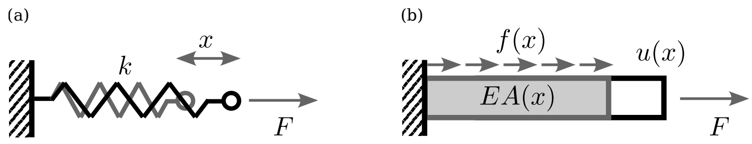
Applying the first boundary condition gives $C_1 = 0$. Enforcing the second boundary condition results in $C_0 = F$, leading to the solution:

$$u\left(x\right) = \frac{F}{EA}x. \tag{2.7}$$

Intuitively, each material point experiences the same internal pulling force and undergoes uniform expansion. The displacement $u(x)$ at position x is the cumulative result of this expansion, producing a linearly increasing function. This result is consistent with expectations. If we evaluate the displacement at the end of the bar and set $EA = k$, we recover the solution for the spring problem:

$$u(1) = \frac{F}{EA} = \frac{F}{k}. \tag{2.8}$$

In these lecture notes, we will discuss such problems more rigorously. It is important to note that the framework presented here is not derived from first principles. Instead, it provides useful models for describing physical phenomena.



$$kx - F = 0$$

$$\frac{d}{dx}\left[EA\frac{du}{dx}\right] + f = 0$$

Figure 2.2: Discrete and continuum mechanics. (a) Spring model. (b) Bar model.

2.2 Fundamental principles of mechanics

We are concerned with how a body deforms when it is subjected to a force. Thus, we need to study: forces, deformations, and their relationship. The study of deformation without considering forces is called *kinematics*, which is purely geometric in nature. The *conservation laws* or *balance laws* govern the study of forces. Finally, the *constitutive relations* establish the link between forces and deformations, completing the analysis.

Chapter 3

Forces and Moments

3.1 Force

In mechanics of materials, we have two different types of forces: a contact force due to a direct contact between two objects and a force between physically separted objects such as electromagnetic and gravitational forces.

Force is the cause of the deformation and motion. It takes a form of a vector; thus, magnitude and direction are required to describe a force. We assume linearity among multiple forces, i.e., more than two forces can be added and considered as a single force when they are applied at the same point (Figure 3.1).

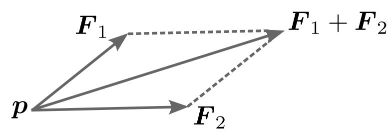


Figure 3.1: Vector summation of forces.

The unit *newton* (N) is defined as that force which gives an acceleration of 1 m/s^2 to a mass of 1 kg.

3.2 Moment of a force

The *moment*, or *torque*, of a force rotates a body about the reference point or, in statics, it twists or bends a body. Let a force $\boldsymbol{F}$ acting on a position $\boldsymbol{p}$ and $\boldsymbol{o}$ denotes a reference point. The displacment vector between $\boldsymbol{o}$ and $\boldsymbol{p}$ is denoted by $\boldsymbol{r} = \boldsymbol{p} - \boldsymbol{o}$. Then, the moment of a force about the point $\boldsymbol{o}$ is defined as (Figure 3.2)

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}. \tag{3.1}$$

In the above, the magnitude of the moment is the magnitude of the force times the distance between $\boldsymbol{o}$ and the line of $\boldsymbol{P}$, i.e., $|\boldsymbol{M}| = h\,|\boldsymbol{F}|$.

The moment $\boldsymbol{M}$ is also a vector quantity perpendicular to the plane determined by $\boldsymbol{r}$ and $\boldsymbol{F}$. Thus, when there is several forces, the total moment about

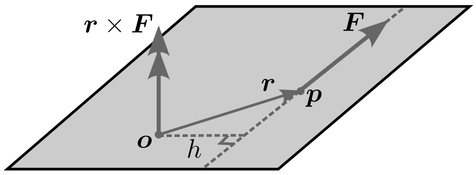


Figure 3.2: Moment of a force.

a point $\boldsymbol{o}$ is obtained as

$$\sum_{i=1}^{N} \boldsymbol{M}_i = \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 + \ldots + \boldsymbol{r}_N \times \boldsymbol{F}_N. \tag{3.2}$$

The corresponding unit is *Newton meter* (Nm).

3.3 Equilibrium

In statics, the *Newton's law of motion* implies that the sum of forces is zero. We say that the forces are *balanced* or are in *equilibrium*.

Given a set of forces acting on different positions, the two necessary conditions for equalibrium are:

$$\mathbf{0} = \sum_{i=1}^{N} \boldsymbol{F}_i = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \ldots + \boldsymbol{F}_N \quad \text{and} \tag{3.3}$$

$$\mathbf{0} = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_i = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 + \ldots + \boldsymbol{r}_N \times \boldsymbol{F}_N. \tag{3.4}$$

In the above, each moment must be defined about the same point.

3.4 Freebody diagram

Freebody diagrams are graphical representations of an object and all forces acting upon on the object (Figure 3.3). Any *supports* can be replaced by reaction forces. In addition, a subset of an object can be considered in separate from the whole object when the influences of the removed parts are replaced by internal forces. Importantly, all forces, translational and rotational, must be balanced for all versions of freebody diagrams.

Determinate structures can be easily analyzed by freebody diagrams. First, we identify all unknowns, i.e., reaction forces, based on the type of supports, or boundary conditions. Second, we calculate reaction forces from the equations we have such as balance of translational and rotational forces. We can determine internal forces by applying the aforementioned process to a specific section of the structure, which is sliced at the desired location.

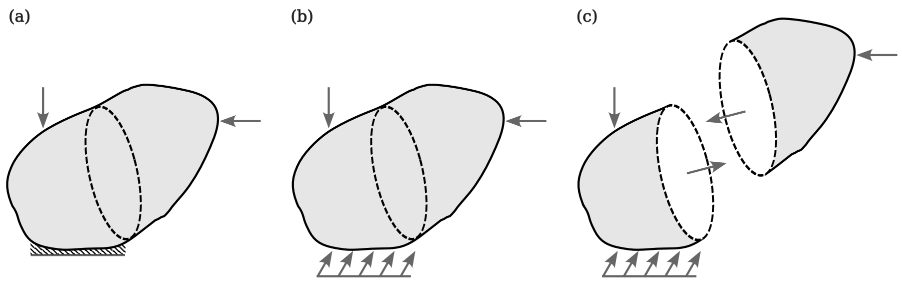


Figure 3.3: Freebody diagrams. (a) A potato under external loads. (b) A support replaced by a reaction force. (c) A sliced potato showing internal forces.

Chapter 4

Stress and Strain

4.1 Stress and Strain in General

Traction, or *stress vector*, is a vector-valued quantity defined on the surface of an object or internally on a cross-section of its subdomain. For example, when a force $\boldsymbol{F}$ is uniformly distributed over an area A, the traction $\boldsymbol{T}$ applied on the surface is given by

$$\boldsymbol{T} = \frac{\boldsymbol{F}}{A}. \tag{4.1}$$

Consequently, traction has units of newtons per square meter (N/m^2), or pascals (Pa). Note that the above expression is valid only for a uniformly distributed force; otherwise, in general (Figure 4.1),

$$\boldsymbol{T} = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{F}}{\Delta A} \quad \text{or} \quad \boldsymbol{F} = \int_A \boldsymbol{T}\, dA. \tag{4.2}$$

A traction vector can be decomposed into three components: one normal to the surface, called the *normal stress*, and two tangential to the surface, called the *shear stresses*.

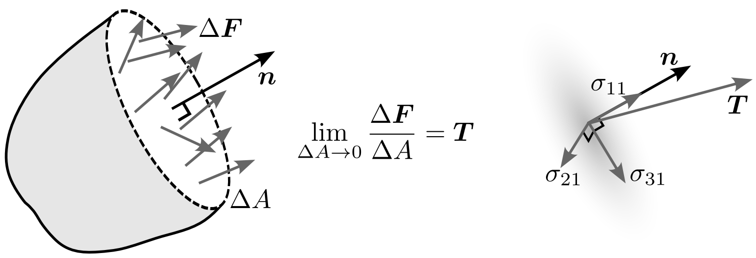


Figure 4.1: Traction and its components.

In general, traction $\boldsymbol{T}$ does not act in the direction of $\boldsymbol{n}$, i.e., the normal vector of the cross-section; however, it does depend on $\boldsymbol{n}$. Therefore, there exists a *stress tensor* $\boldsymbol{\sigma}$ such that

$$\boldsymbol{T}(\boldsymbol{n}) = \boldsymbol{\sigma}\boldsymbol{n} \quad \text{or} \quad \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}. \tag{4.3}$$

Note that the stress tensor is symmetric, i.e., $\sigma_{ij} = \sigma_{ji}$, or $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, and $\sigma_{yz} = \sigma_{zy}$, which is guaranteed by the balance of angular momentum. The existence of the stress tensor is derived from the *balance laws* and *Cauchy's theorem*, which are beyond the scope of these notes.

Strain $\boldsymbol{\varepsilon}$ is another tensor-valued quantity that represents how an infinitesimal element deforms due to stress. More rigorously, it is the spatial rate of displacement $\boldsymbol{u} = (u_x, u_y, u_z)$. In linear elasticity, strain is the symmetric part of the displacement gradient, i.e.,

$$\begin{aligned} \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} &= \operatorname{grad}_{\text{sym.}} \boldsymbol{u} \\ &= \frac{1}{2}\left[(\operatorname{grad} \boldsymbol{u}) + \frac{1}{2}(\operatorname{grad} \boldsymbol{u})^T\right] \\ &= \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \text{sym.} & & \frac{\partial u_z}{\partial z} \end{bmatrix}. \end{aligned} \tag{4.4}$$

Thus, similarly to stresses, strains are symmetric, i.e., $\varepsilon_{ij} = \varepsilon_{ji}$. Note that rigid translations and rotations do not induce stress. Therefore, they are removed from the strain by taking the gradient of the displacement and subsequently extracting its symmetric part.

Stress and strain are related via a constitutive relation. In linear elasticity, this relation is governed by Hooke's law:

$$\boldsymbol{\sigma} = \boldsymbol{C}[\boldsymbol{\varepsilon}] = \boldsymbol{C} : \boldsymbol{\varepsilon} \quad \text{or, in components,} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{4.5}$$

Here, $\boldsymbol{C}$ is the *elasticity tensor*, a rank-four tensor mapping a rank-two tensor to another rank-two tensor. The operation of the rank-four tensor on a rank-two tensor is defined by a *double contraction*, denoted by $:$.

The general elasticity tensor in three dimensions has $3 \times 3 \times 3 \times 3 = 81$ components. However, certain symmetries simplify the relation, resulting in

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} = 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}. \tag{4.6}$$

This relation is characterized by only two parameters, λ and μ, known as the *Lamé parameters*. These parameters can be expressed in terms of a different set (E, ν) as follows:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \tag{4.7}$$

$$\mu = \frac{E}{2(1+\nu)}. \tag{4.8}$$

Here, ν denotes the *Poisson's ratio*, a strictly positive quantity ($0 < \nu < 1$) that measures the extent of lateral expansion or contraction in response to axial loading. For a prismatic bar, Poisson's ratio is defined as (Figure 4.2)

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}}. \tag{4.9}$$

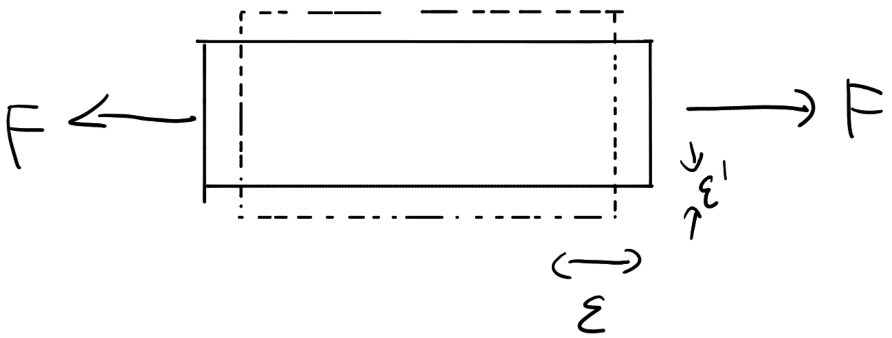


Figure 4.2: Poisson's ratio. ε denotes lateral strain and ε' denotes axial strain.

Another important classification is *isotropy*, which refers to materials exhibiting identical behavior regardless of direction. In contrast, *anisotropy* describes materials whose properties depend on direction. Note that an anisotropic medium may still be homogeneous.

Consider an infinitesimal cubic element subjected to unidirectional tensile loading (Figure 4.3). The initial volume is set to $V = 1$. Assuming isotropy and a uniform Poisson's ratio, the deformed volume V' is given by

$$\begin{aligned} V' &= (1+\varepsilon)(1-\nu\varepsilon)^2 \\ &= (1+\varepsilon)(1-2\nu\varepsilon+\nu^2\varepsilon^2) \\ &= 1-2\nu\varepsilon+\varepsilon+\mathcal{O}(\varepsilon^2). \end{aligned} \tag{4.10}$$

The resulting dilation is measured by

$$\frac{\Delta V}{V} = (1-2\nu)\varepsilon. \tag{4.11}$$

Such a change in volume is fully recovered in elastic deformation but not in plastic deformation.

We will discuss the general properties of stress and strain in more detail later. First, let us familiarize ourselves with special cases through simpler examples.

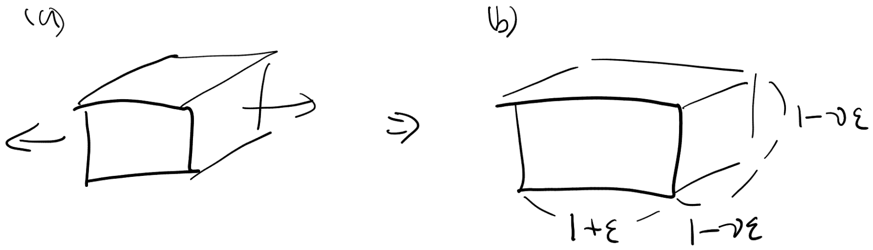


Figure 4.3: Dilation. (a) Before deformation. (b) After deformation.

4.2 Normal Stress and Strain

Consider a prismatic bar subjected to a force applied normal to the centroid of the cross-section (Figure 4.4). The corresponding normal stress is readily obtained as $\sigma = F/A$, which remains constant across cross-sections at different positions x.

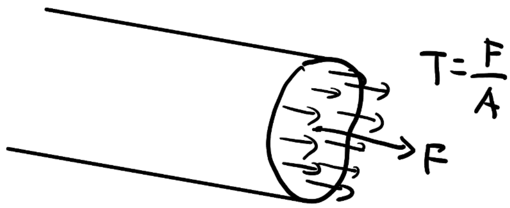


Figure 4.4: Prismatic bar in tension

The standard sign convention for normal stress is to assign a positive value to *tensile stresses* and a negative value to *compressive stresses*. However, alternative conventions may be adopted. For instance, compressive stresses are often considered positive in soil mechanics, as soils exhibit limited resistance to tension.

It is intuitively expected that the total elongation δ is uniformly distributed along x. The elongation per unit length is termed the normal strain, and is given by

$$\epsilon = \frac{\delta}{L}. \tag{4.12}$$

As implied above, strain is a dimensionless quantity.

From the earlier discussion, the elongation was computed using the differential relation $\delta = FL/(EA)$, yielding the strain $\varepsilon = F/(EA)$. This leads to Hooke's law in the form:

$$\sigma = E\varepsilon. \tag{4.13}$$

As mentioned, the uniform normal stress and strain are predicated on the assumption that the force acts through the centroid of the cross-section and in the direction of $\boldsymbol{n}$. A more rigorous statement is that when a uniform normal stress exists, an effective force may be assumed to act at the centroid (x_o, y_o) of the cross-section, where

$$x_o = \frac{\int x\,dA}{A} \quad \text{and} \quad y_o = \frac{\int y\,dA}{A}. \tag{4.14}$$

This expression for the centroid is derived from the internal moment relations:

$$M_x = F y_o = \int \sigma y\,dA, \tag{4.15}$$

$$M_y = F x_o = \int \sigma x\,dA. \tag{4.16}$$

TODO: Computational examples on normal stress and strain;
Poisson's ratio and lateral strain
See examples in [Goodno and Gere, 2017].

4.3 Stress-strain relation

The previous discussion assumed a linear relationship between stress and strain. However, more complex material behavior can be observed experimentally using a tensile testing machine. A specimen is placed between two grips and subjected to tensile loading, while the machine records the resulting deformation (Figure 4.5).

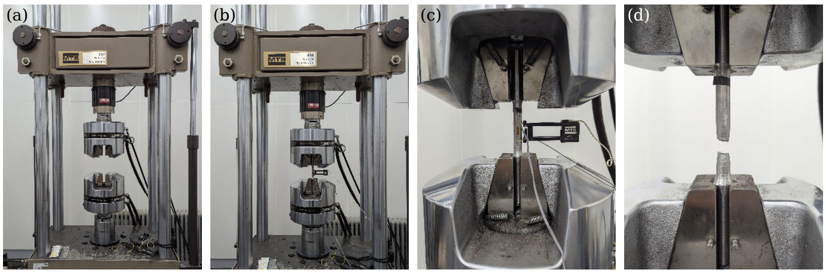


Figure 4.5: Tensile testing machine. (a) Machine. (b) Machine with a sample. (c) Strain meter. (d) After fracture.

Nominal stress is calculated by dividing the applied force by the original cross-sectional area, whereas the *true stress* is defined based on the current, or deformed, cross-sectional area at the point of failure:

$$\sigma_{\text{nominal}} = \frac{F}{A_{\text{initial}}}, \tag{4.17}$$

$$\sigma_{\text{true}} = \frac{F}{A_{\text{current}}}. \tag{4.18}$$

Under a tensile force, a specimen exhibits a noticeable reduction in its cross-sectional area as it approaches the ultimate stress. This phenomenon is referred to as *necking*. As a result, the true stress is generally greater than the nominal stress. Similarly, *nominal strain* is computed by dividing the elongation by the initial gauge length, whereas *true strain* is based on the current, reduced gauge length.

$$\varepsilon_{\text{nominal}} = \frac{\delta}{L_{\text{initial}}}, \tag{4.19}$$

$$\varepsilon_{\text{true}} = \frac{\delta}{L_{\text{current}}}. \tag{4.20}$$

Figure 4.6 illustrates a typical stress-strain relationship. The range from O to A represents the linear region, corresponding to elastic behavior. The stress at point A is called the *proportional limit*, while the stress at point B is referred to as the *yield stress*. Between B and C, the material deforms without an increase in load—this is termed *perfect plasticity* or *yielding*. *Strain hardening* occurs between the yield stress and the *ultimate stress*, caused by changes in the material's crystalline structure. After reaching the ultimate stress, a reduction in applied load is observed up to *failure*, primarily due to necking, while the true stress continues to increase (as indicated by the dashed line).

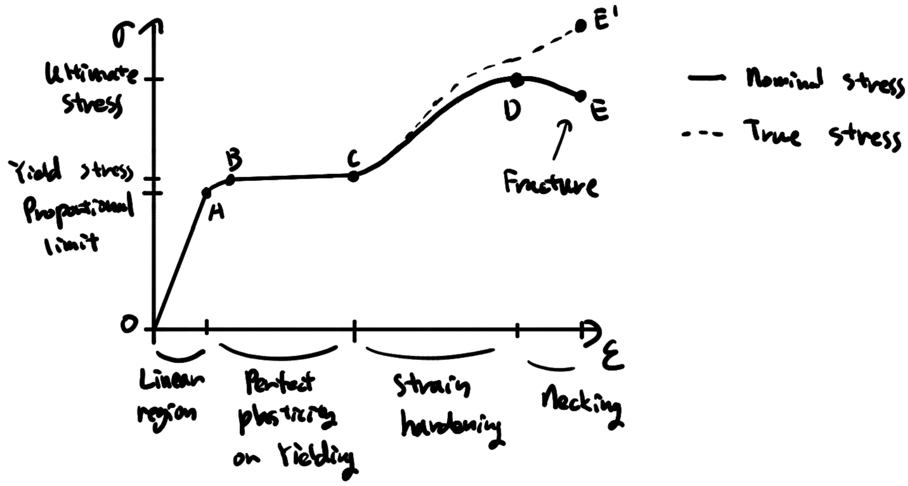


Figure 4.6: Stress-strain diagram (not to scale). Nominal stress is shown as a solid line and true stress as a dashed line.

Elastic recovery is observed when unloading occurs before the specimen reaches the yield stress. In structural design, materials are generally assumed to remain within the elastic region throughout their service life. Unloading beyond the yield point, however, results in permanent deformation, known as *residual strain*.

A material is described as *ductile* if it exhibits a large region of plastic strain (e.g., steel). Conversely, materials such as concrete are generally *brittle*, showing negligible plastic strain before failure (Figure 4.8).

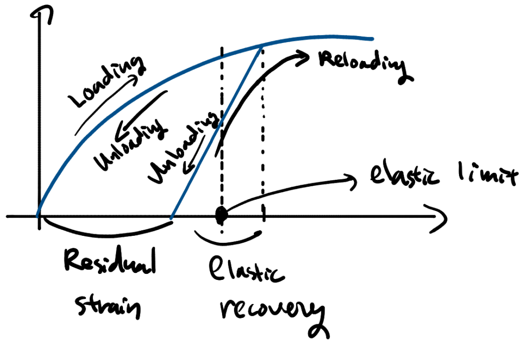


Figure 4.7: Elastic and partially elastic behaviors.

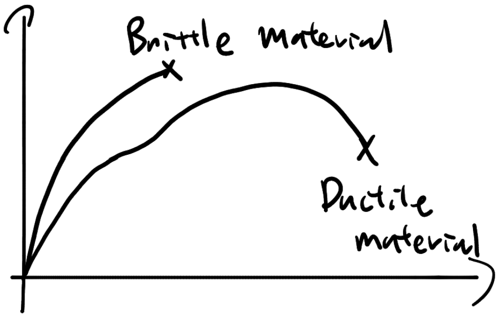


Figure 4.8: Stress-strain curves of brittle and ductile materials.

4.4 Time-dependent Stress-Strain Relation

Strictly speaking, stress-strain responses are not always instantaneous and may not be local in time—although such effects are often neglected in many applications. When time dependency is significant, the constitutive relation can be expressed by a convolution integral, for example:

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \boldsymbol{C}(t)\,[\boldsymbol{\varepsilon}(t)] = \boldsymbol{C}(t) * \boldsymbol{\varepsilon}(t) \\ &= \int_{-\infty}^{t} \boldsymbol{C}(t-\tau) : \boldsymbol{\varepsilon}(\tau)\, d\tau. \end{aligned} \tag{4.21}$$

Thus, the response of stress and strain depends on their entire history. Representative examples include: 1) *Viscoelasticity*, where stress depends on the history of the strain rate; 2) *Creep*, where strain increases under constant stress; and 3) *Relaxation*, where stress decreases under constant strain. Additionally, non-instantaneous responses become necessary in *relativistic elasticity*.

Causality requires $\boldsymbol{C}(t)$ to be single-sided, i.e., $\boldsymbol{C}(t) = 0$ for $t < 0$ (Figure 4.9); hence, the upper bound of the integral (4.21) becomes t. In Figure 4.9(b), the support of the function is approximately limited to the interval $(0, t_o)$. In such cases, the domain of the convolution integral can be approximated by $(t - t_o, t)$.

Further reduction of the support to a single point at $t = 0$, modeled by a *Dirac delta function* $\delta(t)$, implies an instantaneous and local response. This recovers Hooke's law (4.5).

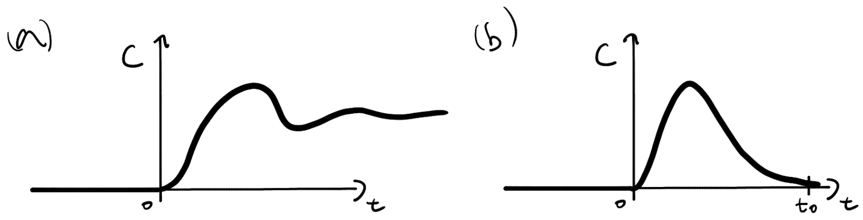


Figure 4.9: Causal (single-sided) functions. (a) General causal function. (b) Causal function with a local support.

4.5 Shear Stress and Strain

Thus far, we have considered normal stress, which acts perpendicular to a surface. In contrast, *shear stress* acts tangentially to a surface.

Figure 4.10 illustrates an example of *single shear*, involving one shearing cross-section. The actual distribution of shear stress across the section is generally difficult to determine analytically. Instead, we use the *average shear stress*, given by:

$$\tau_{\text{average}} = \frac{V}{A}, \tag{4.22}$$

where $V = F$ is the shear force and A is the cross-sectional area.

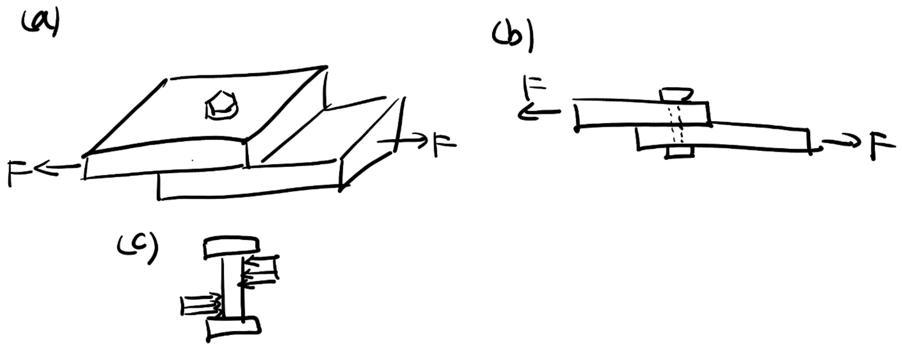


Figure 4.10: Example of single shear. (a) Bolted connection in single shear. (b) Cross-sectional view. (c) Freebody diagram of bolt.

Figure 4.11 presents the case of two shearing cross-sections, known as *double shear*. In this scenario, each surface carries half the load, resulting in

$$\tau_{\text{average}} = \frac{F}{2A}. \tag{4.23}$$

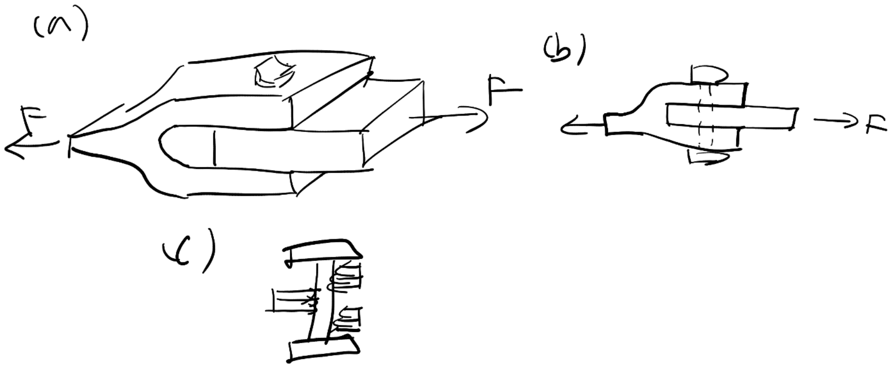


Figure 4.11: Example of double shear. (a) Bolted connection in double shear. (b) Cross-sectional view. (c) Freebody diagram of bolt.

The above two cases are examples of *direct shear*, where the forces act directly to cut through the material. Next, we consider a case of *pure shear* (Figure 4.12). Assuming uniform shear stresses on all sides, we can demonstrate that the shear stresses on opposite faces are equal, i.e., $\tau_1 = \tau_1'$ and $\tau_2 = \tau_2'$. This result follows from the balance of forces:

$$\tau_1 bc - \tau_1' bc = 0. \tag{4.24}$$

Additionally, the balance of moments yields

$$a \cdot \tau_1 bc - c \cdot \tau_2 ab = 0, \tag{4.25}$$

which implies

$$\tau_1 = \tau_2. \tag{4.26}$$

These results imply the symmetry of the stress tensor, i.e., $\sigma_{ij} = \sigma_{ji}$. In summary: 1) Shear stresses on opposite faces of an element are equal in magnitude and opposite in direction; 2) Shear stresses on adjacent faces are equal in magnitude and directed along the shared edge.

Based on these observations, the following sign convention for shear stress is adopted: a shear stress on a positive face is considered positive if it acts in the direction of the coordinate axes *and* a shear stress on a negative face is considered positive if it acts in the direction opposite to the coordinate axes. Here, a positive face is one whose outward normal vector is aligned with the positive direction of the coordinate axes.

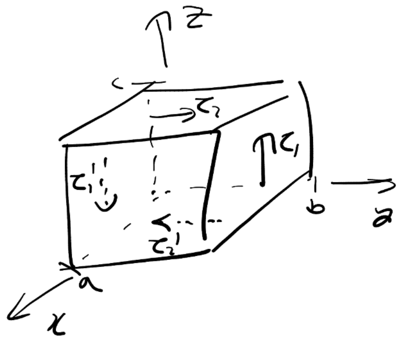


Figure 4.12: Infinitesimal element subjected to pure shear.

The corresponding strain is illustrated in Figure 4.13. The *engineering shear strain* γ measures the angular distortion of the element. It is defined as twice the (tensorial) shear strain:

$$\gamma_{xy} = 2\varepsilon_{xy}. \tag{4.27}$$

The sign convention for shear strain assigns positive values when the angle between positive faces decreases, and negative values otherwise.

Hooke's law for shear stress and strain is given by

$$\tau = G\gamma. \tag{4.28}$$

Here, G is the *shear modulus of elasticity*, which can be expressed in terms of Young's modulus E and Poisson's ratio ν as

$$G = \frac{E}{2(1+\nu)} = \mu. \tag{4.29}$$

TODO: Computational examples on simple and pure shear
See examples in [Goodno and Gere, 2017].

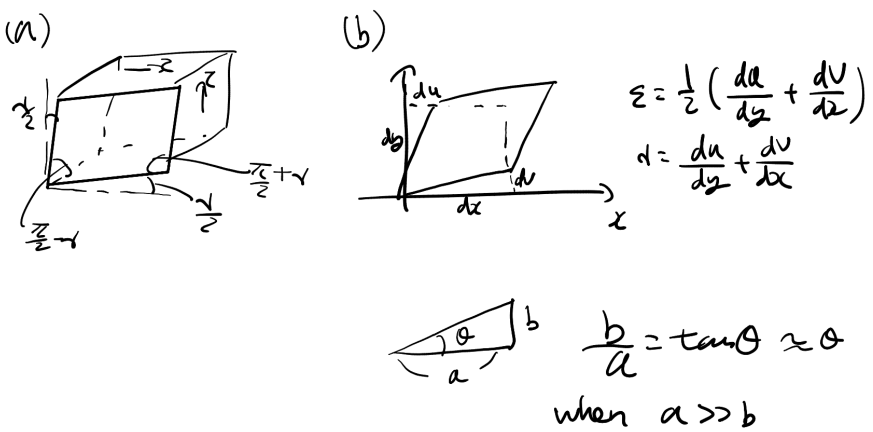


Figure 4.13: Strain due to pure shear. (a) Engineering strain and angular distortion. (b) Engineering strain vs. tensorial strain.

Chapter 5

Axial Load

5.1 Assumptions

We assume *Saint-Venant's principle* [Hibbeler, 2010]:

The stress and strain produced at points in a body sufficiently removed from the region of external load application will be the same as the stress and strain produced by any other applied external loading that has the same statically equivalent resultant and is applied to the body within the same region.

TODO: illustration on Saint-Venant's principle
See corresponding section in [Hibbeler, 2010].

In the scope of axial members, we assume a pure normal stress $\sigma = F/A$, i.e.,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.1}$$

The Hooke's law expressed in terms of Young's modulus E and Poisson's ratio ν reads

$$\boldsymbol{\sigma} = \frac{E}{1+\nu}\left(\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\mathrm{tr}\left(\boldsymbol{\varepsilon}\right)\boldsymbol{I}\right). \tag{5.2}$$

or, inversely,

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{tr}\left(\boldsymbol{\sigma}\right)\boldsymbol{I}. \tag{5.3}$$

Then,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\nu\frac{\sigma}{E} & 0 \\ 0 & 0 & -\nu\frac{\sigma}{E} \end{bmatrix}. \tag{5.4}$$

let $u_x = u\left(x\right)$ depends only on x, we have

$$\sigma = E\varepsilon \quad \text{and} \tag{5.5}$$

$$\varepsilon = \frac{du}{dx}. \tag{5.6}$$

5.2 Governing equation

We now continue our discussion of a prismatic bar subjected to axial loading, but in a more rigorous manner, by starting with the derivation of *bar equation.*

Consider an infinitesimally small element subjected to an external axial load f (Figure 5.1). The balance of force reads

$$0 = \sum F = -F + f\Delta x + F + \Delta F. \tag{5.7}$$

By taking the limit of $\Delta x \to 0$, we have

$$0 = \lim_{\Delta x \to 0} \left(\frac{\Delta F}{\Delta x} + f \right) \Rightarrow \frac{dF}{dx} = -f. \tag{5.8}$$

Thus, we have the relation between the external load f and the internal axial force F.

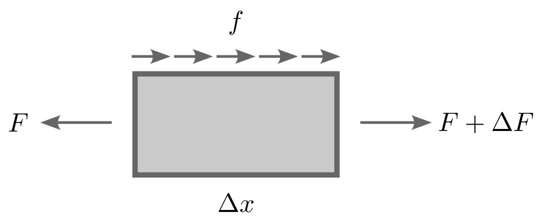


Figure 5.1: Infinitesimally small element for bar equation.

Let u denote the axial displacement; then, the strain-displacement relation is given by

$$\varepsilon = \frac{du}{dx}. \tag{5.9}$$

We assume Hooke's law $\sigma = E\varepsilon$ for the constitutive relation, which gives

$$\sigma = E\frac{du}{dx}. \tag{5.10}$$

Then, we compute the axial force by integrating the normal stress over the cross-section, i.e.,

$$F = \int_A \sigma dA = \int_A E\frac{du}{dx} dA = E\frac{du}{dx}\int_A dA = EA\frac{du}{dx}. \tag{5.11}$$

Thus, we derived the bar equation as

$$\frac{d}{dx}\left[EA\frac{du}{dx}\right] + f = 0. \tag{5.12}$$

The governing equation must be accompanied by boundary conditions. Here, we require two boundary conditions because the highest order of derivative is second. For example, the boundary conditions for a bar with fixed supports at both ends are given by *homogeneous Dirichlet boundary condition* (Figure 5.2(a)), i.e.,

$$u|_{x=0} = 0 \quad \text{and} \quad u|_{x=L} = 0. \tag{5.13}$$

Here, displacement u is constrained to be zero at both ends. The corresponding forces are unknown and should be determined by solving the given problem.

If the fixed support is removed on the right end $x = L$, the force is known to be zero, while the corresponding displacement is unknown. We say that we have *homogeneous Neumann boundary condition* at $x = L$ (Figure 5.2(b)), i.e.,

$$u|_{x=0} = 0 \quad \text{and} \quad EA\left.\frac{du}{dx}\right|_{x=L} = 0. \tag{5.14}$$

Inhomogeneous boundary conditions are possible when a specific value is assigned, such as

$$u = u_o \quad \text{or} \quad EA\frac{du}{dx} = F. \tag{5.15}$$

Figure 5.2: Two types of supports and boundary conditions. (a) Fixed ends at $x = 0$ and $x = L$ (homogeneous Dirichlet condition at both boundaries). (b) Free end at $x = L$ (homogeneous Neumann condition at the right boundary).

For example, consider a bar with fixed ends subjected to a distributed load f, i.e.,
Given a constant axial rigidity EA and an external load $f(x) = f_o x (L - x)/L^2$, find the displacement $u(x)$ such that

$$\frac{d}{dx}\left[EA\frac{du}{dx}\right] + f = 0, \quad x \in (0, L), \tag{5.16}$$

$$u(0) = u(L) = 0. \tag{5.17}$$

Integrating the governing equation, we have

$$EA\frac{du}{dx} = \frac{f_o}{L^2}\frac{1}{3}x^3 - \frac{f_o}{L^2}\frac{1}{2}x^2 L + C_1 \quad \text{and} \tag{5.18}$$

$$EAu = \frac{f_o}{L^2}\frac{1}{12}x^4 - \frac{f_o}{L^2}\frac{1}{6}x^3 L + C_1 x + C_2. \tag{5.19}$$

Then, the two boundary conditions give

$$u(0) = 0 \Rightarrow C_2 = 0 \quad \text{and} \tag{5.20}$$

$$u(L) = 0 \Rightarrow f_o L^2\left(\frac{1}{12} - \frac{1}{6}\right) = -C_1 L \Rightarrow C_1 = \frac{f_o L}{12}. \tag{5.21}$$

Thus, we have

$$u = \frac{f_o}{12EAL^2}\left(x^4 - 2Lx^3 + L^2x\right) \quad (5.22)$$

or

$$\frac{u}{L} = \frac{f_oL}{12EA}\left[\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)\right]. \quad (5.23)$$

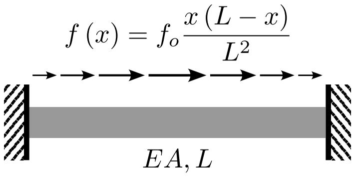


Figure 5.3: Bar with fixed ends subject to a spatially varying external load.

5.3 Statically determinate structure

We say that a structure is statically determinate when a problem can be analyzed by freebody diagrams. Namely, the reaction forces are fully determined by investigating equilibrium equations of a freebody diagram without a need of solving differential equation.

Consider a bar with a fixed end at $x = 0$ and a free end at $x = L = L_1 + L_2 + L_3$. The bar is subjected to three external loads $-F_B$, F_C, and F_D, respectively at $x = L_1$, $x = L_1 + L_2$, and $x = L$.

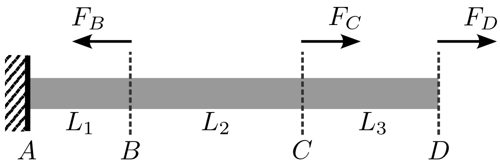


Figure 5.4: Statically determinate bar with three point loads.

The reaction force R_A at $x = 0$ is calculated by

$$0 = \sum F = -R_A - F_B + F_C + F_D \Rightarrow R_A = F_C + F_D - F_B. \quad (5.24)$$

Repeating the same analysis for freebody diagrams with different cross-sectional locations (Figure 5.5), we have

$$F_1 = F_C + F_D - F_B, \quad (5.25)$$
$$F_2 = F_C + F_D, \quad \text{and} \quad (5.26)$$
$$F_3 = F_D. \quad (5.27)$$

Then, corresponding elongation at each segment reads

$$\delta_1 = \frac{F_1 L_1}{EA_1} = (F_C + F_D - F_B)\,\frac{L_1}{EA_1}, \tag{5.28}$$

$$\delta_2 = \frac{F_2 L_2}{EA_2} = (F_C + F_D)\,\frac{L_2}{EA_2}, \quad \text{and} \tag{5.29}$$

$$\delta_3 = \frac{F_3 L_3}{EA_3} = F_D\,\frac{L_3}{EA_3}, \tag{5.30}$$

where the total elongation is $\delta = \delta_1 + \delta_2 + \delta_3$.

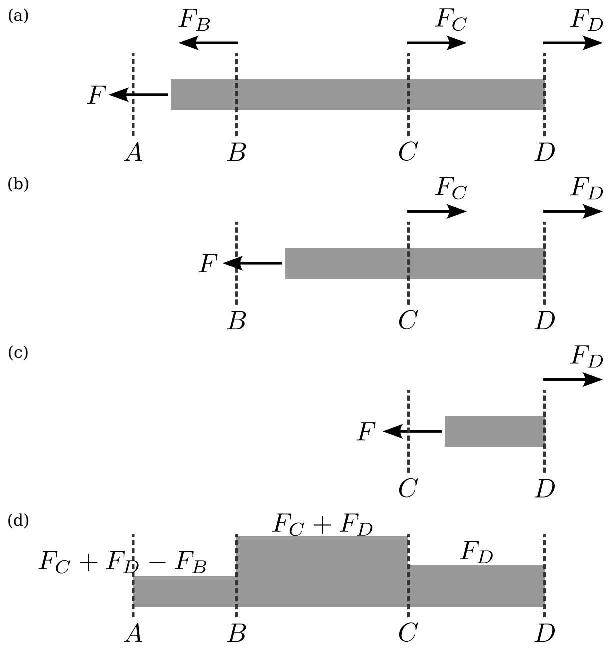


Figure 5.5: Freebody diagrams and axial force diagram. (a),(b),(c) Freebody diagrams with different cross-sections for calculating axial forces. (d) Axial force diagram.

Next, consider a case with spatially varying axial rigidity $EA\,(x)$. $u = 0$ at $x = 0$ and $EA(du/dx) = F$ at $x = L$. From the equilibrium equation, we have

$$R_A = -F. \tag{5.31}$$

Here, we have a constant axial force throughout the domain. Then, from the relation $F = EA(du/dx)$, we have

$$u\,(x) = \int_0^x \frac{F}{EA\,(\xi)} d\xi. \tag{5.32}$$

5.4 Statically indeterminate structure

A *statically indeterminate structure* is a structure that cannot be analyzed by free body diagrams, i.e., by equilibrium equations. Such a problem occurs when the number of unknowns is greater than the number of equilibrium equations. Thus, we need additional equations to determine a *compatibility condition*.

Consider an example shown in Figure 5.6. Here, we have two axial members of different EA and L attached to each other. The two ends are fixed, i.e., $u(0) = u(L) = 0$, $L = L_1 + L_2$. Force F is applied at the interface between the two members. We identify two reaction forces, R_A and R_B, when the two supports are removed (Figure 5.6(b)). The corresponding equilibrium equation reads

$$-R_A + F + R_B = 0. \tag{5.33}$$

Here, we have two unknowns but only one equation; thus, the problem is underdetermined or statically indeterminate. We may try a different version of the free body diagram as shown in Figure 5.6(c). We have three equilibrium equations:

$$\begin{cases} -R_A + F_C^L = 0 \\ -F_C^L + F + F_C^R = 0 \\ -F_C^R + R_B = 0 \end{cases} \tag{5.34}$$

The above problem is still not solvable because we have four unknowns, R_A, R_B, F_C^L, and F_C^R. Again, we are short of one equation.

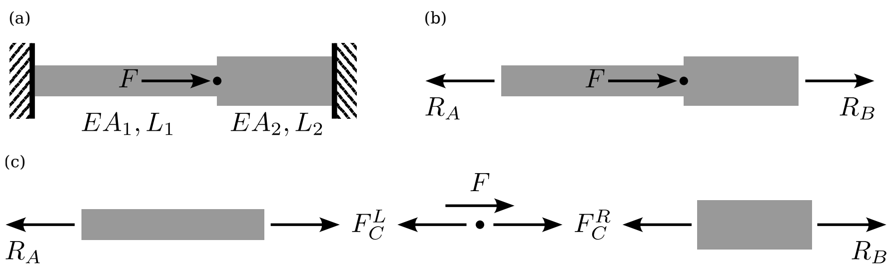


Figure 5.6: Statically indeterminate structure. (a) Structure and forces. (b) Freebody diagram when supports are removed. (c) Freebody diagram when two members are separated.

However, we know that the total length of the structure must be remained unchanged due to the given boundary conditions. Note that each member may deform due to the internal forces. Then, we can state that the sum of elongation of two members is zero, which gives the compatibility condition:

$$\delta_1 + \delta_2 = 0. \tag{5.35}$$

Here, the elongation of each member is denoted by δ_1 and δ_2, respectively. The internal forces at each member are constant by R_A and R_B, respectively; (5.35) becomes

$$\frac{R_A L_1}{EA_1} + \frac{R_B L_2}{EA_2} = 0. \tag{5.36}$$

Then, we have

$$R_A = F\frac{\frac{L_1}{EA_1}}{\frac{L_1}{EA_1}+\frac{L_2}{EA_2}} \quad \text{and} \tag{5.37}$$

$$R_B = -F\frac{\frac{L_2}{EA_2}}{\frac{L_1}{EA_1}+\frac{L_2}{EA_2}}. \tag{5.38}$$

Next, consider two members being pulled by a force F, where one end is fixed and the other end are constrained by a rigid plate so that the elongation of the two members become identical (Figure 5.7). Here we have two four unknowns and three equations:

$$\begin{cases} -R_A^1 + R_B^1 = 0 \\ -R_A^2 + R_B^2 = 0 \\ -R_B^1 - R_B^2 + F = 0 \end{cases}. \tag{5.39}$$

Thus, the given structure is statically indeterminate. As implied above, the corresponding compatibility equation is

$$\delta_1 = \delta_2 \tag{5.40}$$

or

$$\frac{R_A^1 L}{EA_1} = \frac{R_A^2 L}{EA_2}. \tag{5.41}$$

Then, we have

$$R_A^1 = R_B^1 = F\frac{EA_1}{EA_1+EA_2} \quad \text{and} \tag{5.42}$$

$$R_A^2 = R_B^2 = F\frac{EA_2}{EA_1+EA_2}. \tag{5.43}$$

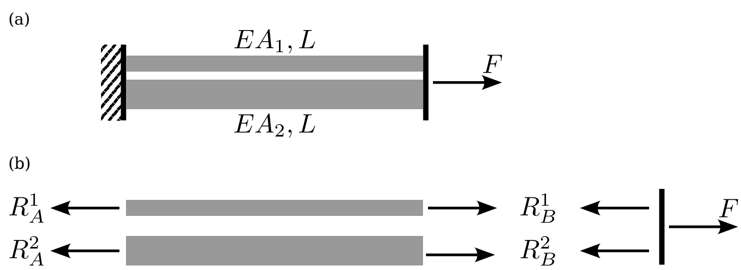


Figure 5.7: Another example of statically indeterminate structure. (a) Structure and forces. (b) Freebody diagram.

5.5 Thermal effects and misfits

Here, we consider a uniform temperature change within a member denoted by ΔT. Then, the strain due to the temperature is given by

$$\varepsilon_T = \alpha\,(\Delta T)\,, \tag{5.44}$$

where α is the coefficient of thermal expansion. Then, we have $\delta_T = \varepsilon_T L$ when the length of the member is L. Statically determinate structures will freely expand without inducing internal forces. However, thermal expansion may constrained for indeterminate structures and causes internal forces.

For example, consider a case shown in Figure 5.8. Here, the structure is constrained so that the total elongation is zero. The compatibility equation reads

$$\delta_T + \delta_S = 0, \tag{5.45}$$

where $\delta_T = \alpha (\Delta T) L$ is the thermal expansion and $\delta_S = FL/(EA)$ is the elongation due to the internal force F. Then, we have

$$F = -\alpha (\Delta T) EA. \tag{5.46}$$

The corresponding reaction forces at two ends has the same magnitude above in the directions of compression.

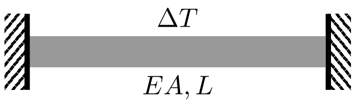


Figure 5.8: Uniform temperature change.

Misfits, or fabrication errors, are treated the same way as the thermal effects with given amount of strain $\varepsilon_E = \delta_E/L$, where δ_E is the fabrication error in length.

Chapter 6

Torsion

6.1 Assumptions

Torsional member is easier to describe in cylindrical coordinates (ρ, θ, x):

$$\begin{cases} x = x \\ y = \rho \cos\theta \\ z = \rho \sin\theta \end{cases} . \tag{6.1}$$

In addition to the small strain assumption and Saint-Venant's principle, we assume

$$\begin{aligned} u_\rho &= 0, && (6.2)\\ u_\theta &= \varphi(x)\,\rho, \quad \text{and} && (6.3)\\ u_x &= 0. && (6.4) \end{aligned}$$

Then, the strain components reads

$$\begin{aligned} \boldsymbol{\varepsilon} &= \frac{1}{2}\left[(\operatorname{grad}\boldsymbol{u}) + (\operatorname{grad}\boldsymbol{u})^T\right] \\ &= \begin{bmatrix} \frac{\partial u_\rho}{\partial \rho} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\rho}{\partial \theta} + \frac{\partial u_\theta}{\partial \rho} - \frac{u_\theta}{\rho}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial \rho} + \frac{\partial u_\rho}{\partial x}\right) \\ & \frac{1}{\rho}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\rho}{\rho} & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial x} + \frac{1}{\rho}\frac{\partial u_z}{\partial \theta}\right) \\ \text{sym.} & & \frac{\partial u_x}{\partial x} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\rho\frac{d\varphi}{dx} \\ 0 & \frac{1}{2}\rho\frac{d\varphi}{dx} & 0 \end{bmatrix} \end{aligned} \tag{6.5}$$

Then, from the Hooke's law, we identify a pure shear, i.e.,

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{C} : \boldsymbol{\varepsilon} = 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\,\boldsymbol{I} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu\rho\frac{d\varphi}{dx} \\ 0 & \mu\rho\frac{d\varphi}{dx} & 0 \end{bmatrix}. \end{aligned} \tag{6.6}$$

In the above, $\mu = G$ is the shear modulus.

TODO: three-dimensional figure showing shear stress (pure shear)
See [Hibbeler, 2010].

6.2 Governing equation

We consider a one-dimensional member subjected to a *twisting moment*, or *torque*. Thus, we are interested in finding the angle of rotation $\varphi(x)$ as opposed to axial displacement $u(x)$ due to axial forces (Figure 6.1).

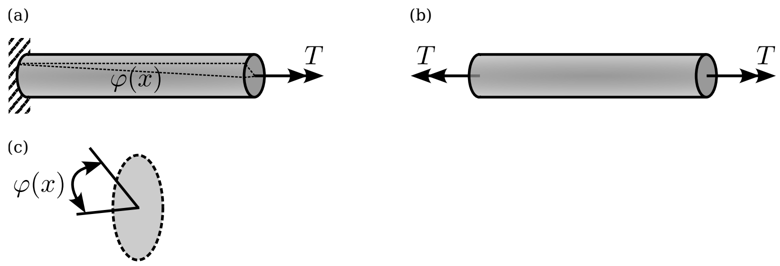


Figure 6.1: Deformation of a circular bar in pure torsion. (a) Structure and torsion. (b) Freebody diagram. (c) Angle of rotation.

Consider an infinitesimally small element subjected to an external distributed moment m (Figure 6.2). The balance of moment reads

$$0 = \sum M = -T + m\Delta x + T + \Delta T. \tag{6.7}$$

By taking the limit of $\Delta x \to 0$, we have

$$0 = \lim_{\Delta x \to 0} \left(\frac{\Delta T}{\Delta x} + m \right) \Rightarrow \frac{dT}{dx} = -m. \tag{6.8}$$

Thus, we have the relation between the distributed moment m and the internal twisting moment T.

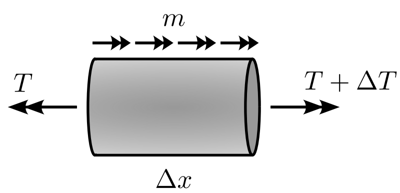


Figure 6.2: Infinitesimally small element for torsion.

Let φ denote the angle of rotation; then, the strain-displacement relation is given by

$$\gamma = \rho \frac{d\varphi}{dx}. \tag{6.9}$$

Here, ρ is the distance between a position in a cross-section and its axis and γ is an engineering shear strain. We assume Hooke's law $\tau = G\varphi$ for the constitutive relation, which gives

$$\tau = G\rho \frac{d\varphi}{dx}. \tag{6.10}$$

Then, we compute the twisting moment by integrating the shear stress over the cross-section, i.e.,

$$T = \int_A \tau \rho dA = \int_A G\rho^2 \frac{d\varphi}{dx} dA = G\frac{d\varphi}{dx}\int_A \rho^2 dA = GJ\frac{d\varphi}{dx}. \tag{6.11}$$

Here, J is called *the second polar moment of area.* Thus, we derived the equation for torsion as

$$\frac{d}{dx}\left[GJ\frac{d\varphi}{dx}\right] + m = 0. \tag{6.12}$$

The corresponding Dirichlet and Neumann boundary conditions read, respectively,

$$\varphi = \varphi_0 \quad \text{and} \tag{6.13}$$

$$GJ\frac{d\varphi}{dx} = T. \tag{6.14}$$

6.3 Pure torsion

Let us revisit the simple case of pure torsion, i.e.,

$$\begin{cases} \dfrac{d}{dx}\left[GJ\dfrac{d\varphi}{dx}\right] = 0 & x \in (0, L) \\ \varphi = 0 & x = 0 \\ GJ\dfrac{d\varphi}{dx} = T & x = L \end{cases}. \tag{6.15}$$

Here, we consider a constant GJ.

Integrating the governing equation, we have

$$GJ\frac{d\varphi}{dx} = C_1 \tag{6.16}$$

$$GJ\varphi = C_1 x + C_2. \tag{6.17}$$

Then, we determine the two constants from the boundary conditions as

$$C_1 = T \quad \text{and} \quad C_2 = 0. \tag{6.18}$$

Thus, we have the solution

$$\varphi(x) = \frac{T}{GJ}x. \tag{6.19}$$

The total angle of rotation is given by

$$\phi = \frac{TL}{GJ}. \tag{6.20}$$

The above expression implies that the displacement is inversely proportional to J, which represents a cross-sectional character. Below, we consider two different cross-sections: (a) circle and (b) circular tube (Figure 6.3).

The surface element in polar coordinate is $dA = \rho d\rho d\theta$. Then, we have

$$J_{\text{circle}} = \int_A \rho^2 dA = \int_0^{2\pi} \int_0^{\rho_2} \rho^3 d\rho d\theta = \frac{\pi}{2}\rho_2^4 \quad \text{and} \tag{6.21}$$

$$J_{\text{tube}} = \int_A \rho^2 dA = \int_0^{2\pi} \int_{\rho_1}^{\rho_2} \rho^3 d\rho d\theta = \frac{\pi}{2}\left(\rho_2^4 - \rho_1^4\right). = \frac{\pi}{2}\rho_2^4\left(1 - r^4\right). \tag{6.22}$$

Here, $r = \rho_1/\rho_2 < 1$. Thus, J_{circle} and J_{tube} become similar because $r^4 \ll 1$. For example when $r = 0.5$, we have

$$\frac{J_{\text{tube}}}{J_{\text{circle}}} \approx 0.94. \tag{6.23}$$

Thus, a thin tube with a larger radius is more economic, uses less material, than a circular cross-section.

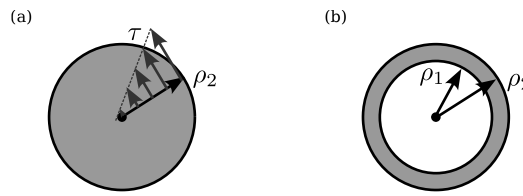


Figure 6.3: Two cross-sections. (a) Circle. (b) Circular tube.

6.4 Statically indeterminate structure

Similarly to axial load, we could consider determinate and indeterminate structures under torsion. Here, we seek compatibility conditions in terms of the angle of rotation.

Consider a bar with fixed ends subjected to two twisting moments (Figure 6.4(a)). Here, lengths of tree subdomains are L_1, L_2, and L_3, respectively. The corresponding angle of rotations are respectively denoted by ϕ_1, ϕ_2, and ϕ_3. Then, we have the equilibrium equations as

$$\begin{cases} -T_A + T_C^L = 0 \\ -T_C^L - T_L + T_C^R = 0 \\ -T_C^R + T_D^L = 0 \\ -T_D^L + T_R + T_D^R = 0 \\ -T_D^R + T_B = 0 \end{cases} \tag{6.24}$$

and the compatibility equation as

$$\phi_1 + \phi_2 + \phi_3 = 0. \tag{6.25}$$

For each subdomain, the twisting moment is constant. Thus, we have

$$\phi_1 = \frac{T_1 L_1}{GJ}, \tag{6.26}$$

$$\phi_2 = \frac{T_2 L_2}{GJ}, \quad \text{and} \tag{6.27}$$

$$\phi_3 = \frac{T_3 L_3}{GJ}, \tag{6.28}$$

where, from the equilibrium equation, $T_1 = T_A$, $T_2 = T_A + T_L$, and $T_3 = T_A + T_L - T_R$. Next, we plug the above rotation angle into the compatibility equation, i.e.,

$$\frac{1}{GJ}\left[T_A L_1 + (T_A + T_L)\, L_2 + (T_A + T_L - T_R)\, L_3\right] = 0 \tag{6.29}$$

or

$$T_A = \frac{T_R L_3 - T_L\,(L_2 + L_3)}{L_1 + L_2 + L_3} \quad \text{and} \tag{6.30}$$

$$T_B = \frac{T_L L_1 - T_R\,(L_1 + L_2)}{L_1 + L_2 + L_3}. \tag{6.31}$$

One possible twisting moment diagram is shown in Figure 6.4(d) when $2L_1 = 2L_2 = L_3 = 2$ and $2T_L = T_R = 2$, which gives

$$T_1 = \frac{1}{2},\ T_2 = \frac{3}{2},\ \text{and } T_3 = -\frac{1}{2}. \tag{6.32}$$

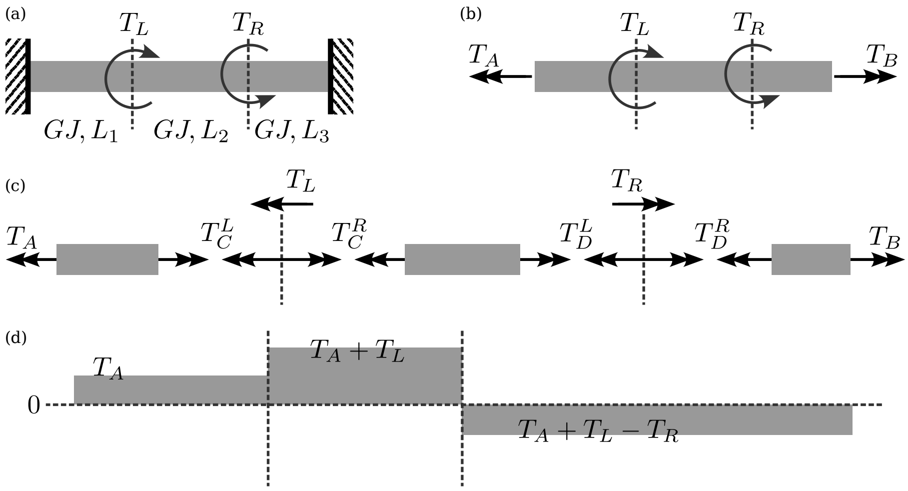


Figure 6.4: Bar with fixed ends subjected to twisting moments. (a) Structure and external moments. (b) Freebody diagram when supports are removed. (c) Freebody diagram showing internal torsional moments. (d) Twisting moment diagram when $2L_1 = 2L_2 = L_3 = 2$ and $2T_L = T_R = 2$.

Consider another example shown in Figure 6.5. Here, the bar is composed with two different materials and, therefore, different torsional rigidity $G_i J_i$, $i = 1, 2$. The total twisting moment T is divided by

$$T = T_1 + T_2, \tag{6.33}$$

where T_1 acts on the inner core and T_2 acts on the shell. We assume the angles of rotation are identical between the two materials, which gives the compatibility equation as

$$\phi_1 = \phi_2. \tag{6.34}$$

or

$$\frac{T_1 L}{G_1 J_1} = \frac{T_2 L}{G_2 J_2}. \tag{6.35}$$

Then, we have

$$T_1 = \frac{T G_1 J_1}{G_1 J_1 + G_2 J_2} \quad \text{and} \tag{6.36}$$

$$T_2 = \frac{T G_2 J_2}{G_1 J_1 + G_2 J_2}. \tag{6.37}$$

The corresponding shear strain is computed as

$$\gamma = \rho \frac{d\varphi}{dx} = \rho \frac{Tx}{G_1 J_1 + G_2 J_2}. \tag{6.38}$$

Let the radii of the inner core and the outer shell are given by a and b. Then, the stresses of the core and the shell are

$$\tau_1 = G_1 \rho \frac{Tx}{G_1 J_1 + G_2 J_2}, \quad 0 < \rho < a \quad \text{and} \tag{6.39}$$

$$\tau_2 = G_2 \rho \frac{Tx}{G_1 J_1 + G_2 J_2}, \quad a < \rho < b. \tag{6.40}$$

Thus, the shear stress has a jump at the interface $\rho = a$ by

$$\tau_2 - \tau_1 = (G_2 - G_1) \frac{Ta}{G_1 J_1 + G_2 J_2}. \tag{6.41}$$

However, this result does not mean that stress continuity at the interface is violated. Note that the continuity must hold across the interface not along the interface.

TODO: provide figures for the continuity problem.

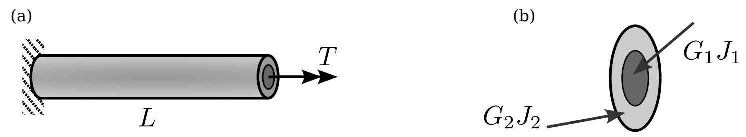


Figure 6.5: Bar with fixed-free end subjected to a twisting moment. (a) Structure and external moments. (b) Cross-section and torsional rigidity GJ.

Chapter 7

Bending

7.1 Assumptions

In addition to small-strain assumption and Saint-Venant's principle, we adopt the assumptions of the *Euler-Bernoulli beam theory*:

- a plane section remains plane after deformation;
- the normal to the plane remains normal; and
- the vertical displacement is uniform across the beam's depth.

The first and second assumptions imply no shear deformation; thus, no distortion of cross-section. The last assumption implies that there is no Poisson's effect, i.e., $\nu = 0$.

The corresponding displacement field reads

$$u_x = -\frac{du_y}{dx}y, \tag{7.1}$$

$$u_y = w(x), \quad \text{and} \tag{7.2}$$

$$u_z = 0. \tag{7.3}$$

In defining u_x, y is measured from the neutral axis such that $\int_A \sigma_{xx} dA = 0$. Then, from the strain-displacement relation, we have

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \text{sym.} & & \frac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} -\frac{d^2 w}{dx^2}y & 0 & 0 \\ & 0 & 0 \\ \text{sym.} & & 0 \end{bmatrix}. \tag{7.4}$$

Then, the only nonzero component of the stress tensor is given by

$$\sigma_{xx} = E\varepsilon_{xx} = -E\frac{d^2 w}{dx^2}y. \tag{7.5}$$

Unfortunately, the above result violates momentum equlibrium at the stress level, i.e.,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0. \tag{7.6}$$

Here, $\sigma_{xz} = 0$ due to the absence of strain or constraints in the z direction. Then, this implies that σ_{xy} must be nonzero to statisfy the above local equilibrium. However, at the macroscopic level, force equilibrium is satisfied when the stress components are integrated over the cross-section. Thus, in the absence of axial force, we require

$$\int_A \sigma_{xx}\, dA = 0. \tag{7.7}$$

The above determines the location of neutral axis by

$$0 = \int_A \sigma_{xx} dA = \int_A -E\frac{d^2 w}{dx^2} y dA \Rightarrow \int_A y dA = 0. \tag{7.8}$$

The balance of momentum in statics reads

$$\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0},$$

or, in Cartesian,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = 0,$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = 0,$$
$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0.$$

We will subsequently "correct" the omitted shear stress σ_{xy} a posteriori using the moment equilibrium.

TODO: a figure showing linear normal stress on a cross-section.

7.2 Governing equation

We derive the beam equation from the equilibrium equations of an infinitesimal element (Figure 7.1). First, we identify the blance of forces and balance of moment of forces:

$$0 = \sum F_y = V - (V + \Delta V) + q \cdot \Delta x \quad \text{and} \tag{7.9a}$$

$$0 = \sum M_R = -M + (M + \Delta M) - V \cdot \Delta x - \frac{1}{2} q \left(\Delta x\right)^2. \tag{7.9b}$$

Then, we take the limit of $\Delta x \to 0$ that keeps linear terms only

$$0 = \lim_{\Delta x \to 0} \left(\frac{\Delta V}{\Delta x} - q \right) \Rightarrow \frac{dV}{dx} = q \quad \text{and} \tag{7.10a}$$

$$0 = \lim_{\Delta x \to 0} \left(\frac{\Delta M}{\Delta x} - V - \frac{1}{2} q \Delta x \right) \Rightarrow \frac{dM}{dx} = V. \tag{7.10b}$$

Thus, the above result gives the relation between external load and shear forces and bending moment, i.e.,

$$\frac{d^2 M}{dx^2} = \frac{dV}{dx} = q. \tag{7.11}$$

Next, we consider geometrical quantities $w = u_y$, $u = u_x$, and θ, under deformation. Figure 7.2 shows how the angle θ is related to the normal displacement

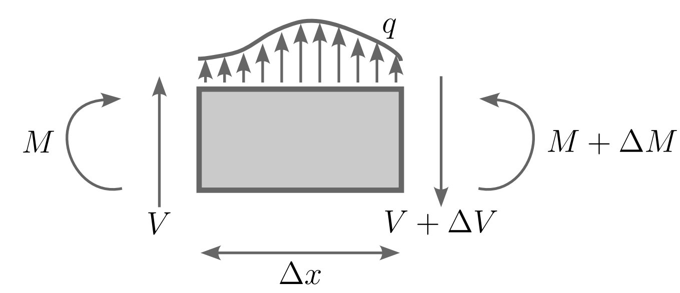


Figure 7.1: Infinitesimal beam element.

u and deflection w. Under a small deformation, we assume that the cross-section rotates but remains flat; then, we have the following strain-displacement relation:

$$\left.\begin{aligned} &\lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = \frac{dw}{dx} = \tan\theta\ (\approx \theta) \\ &u = -y\tan\theta = -\frac{dw}{dx}y \end{aligned}\right\} \Rightarrow \varepsilon = \frac{du}{dx} = -\frac{d^2w}{dx^2}y. \tag{7.12}$$

In the above, ε is the normal strain.

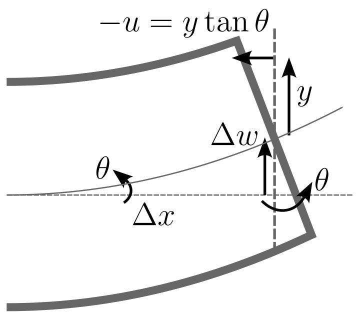


Figure 7.2: Geometrical quantities under deformation.

Then, the final piece we need is the relation between the geometrical quantities and the forces, i.e., the constitutive relation. Here, we use the Hooke's law:

$$\sigma = E\varepsilon = -E\frac{d^2w}{dx^2}y. \tag{7.13}$$

In the above, σ is the normal stress at the cross-section, where the definition of moment states

$$M = -\int_A \sigma y dA = -\int_A E\varepsilon y dA = \int_A E\frac{d^2w}{dx^2}y^2 dA = EI\frac{d^2w}{dx^2}. \tag{7.14}$$

Here, we introduced the *second moment of area* I that captures the cross-sectional information, i.e.,

$$I = \int_A y^2 dA. \tag{7.15}$$

Here, y is measured from the neutral axis of the cross-section. Then, we have derived the beam equation:

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = q. \tag{7.16}$$

The above beam equation is a fourth-order differential equation. Thus, the Dirichlet boundary conditions involve with deflection w and its derivative, i.e., slope, dw/dx, while the Neumann boundary conditions involve with moment and shear force, i.e, EId^2w/dx^2 and $(d/dx)(EId^2w/dx^2)$. We require four boundary conditions to determine the solution $w(x)$.x

In summary, we have

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] - q = 0, \quad x \in (0, L), \tag{7.17}$$

where w is the deflection, q is the distributed load per unit length, and I the second moment of area. The derivatives of w represents the following physical quantities (for a small w):

$$\theta = \frac{dw}{dx} \quad \text{(slope or deflection angle)}, \tag{7.18}$$

$$M = EI\frac{d^2w}{dx^2} \quad \text{(bending moment), and} \tag{7.19}$$

$$V = \frac{d}{dx}\left[EI\frac{d^2w}{dx^2}\right] \quad \text{(shear force)}. \tag{7.20}$$

The above equations are based on the right-handed coordinate system; for the left-handed coordinate system, the governing equation remains the same but moment and shear force are described by $M = -EId^2w/dx^2$ and $V = -(d/dx)(EIdw^2/dx^2)$.

7.3 Supports and boundary condition

Various types of supports are translated in different boundary conditions (Figure 7.3). Common support types and their characteristics are:

- Fixed support: all motions, horizontal, vertical, and rotational, are constrained. Thus, we expect horizontal, vertical, and moment reaction forces.
- Hinge support: horizontal and vertical motions are constrained but rotational motions are allowed. We expect horizontal and vertical reaction forces.
- Roller support: vertical motions are constrained, while other motions are allowed. We expect a vertical reaction force only.

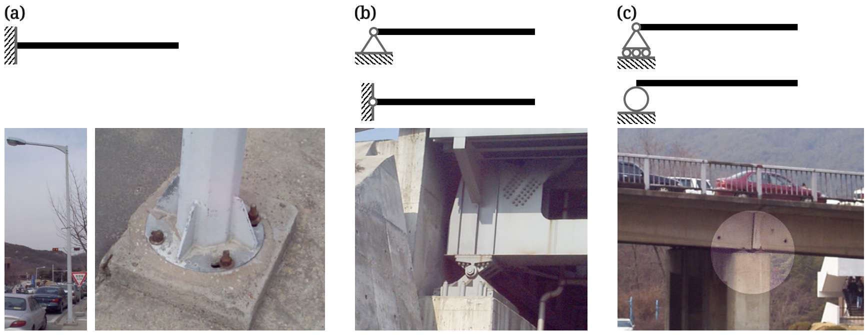


Figure 7.3: Three types of support. (a) Fixed support. (b) Hinge support. (c) Roller support (bridge over a tennis court at Seoul National University). Images courtesy of Hae Sung Lee.

7.4 Sign Convention

The right-handed coordinate system is the most commonly used convention. However, the left-handed coordinate system can also be useful, particularly when assigning positive values to downward displacements, as typically occurs under gravitational loading.

Sign conventions for internal forces may also differ. In these notes, we adopt the convention illustrated in Figure 7.4.

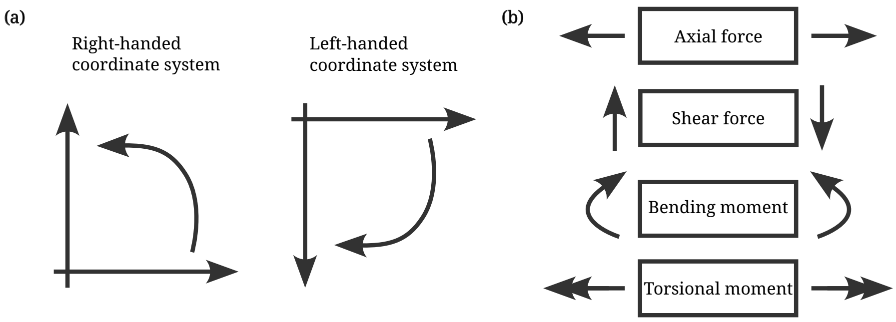


Figure 7.4: Sign conventions. (a) Coordinate systems. (b) Internal forces acting on an infinitesimally small element. Arrows indicate positive directions.

7.5 Cantilever beam

Consider a cantilever beam (Figure 7.5) subject to a uniform distributed load $-q$ (downward):

$$\begin{cases} \frac{d^2}{dx^2}\left[EI\frac{d^x w}{dx^2}\right] = -q & x \in (0, L) \\ w = 0 & x = 0 \\ \frac{dw}{dx} = 0 & x = 0 \\ EI\frac{d^2 w}{dx^2} = 0 & x = L \\ \frac{d}{dx}\left[EI\frac{d^2 w}{dx^2}\right] = 0 & x = L \end{cases} . \tag{7.21}$$

Let $EI =$ constant, we have

$$\frac{d}{dx}\left[EI\frac{d^2 w}{dx^2}\right] = -qx + C_1, \tag{7.22}$$

$$EI\frac{d^2 w}{dx^2} = -\frac{q}{2}x^2 + C_1 x + C_2, \tag{7.23}$$

$$EI\frac{dw}{dx} = -\frac{q}{6}x^3 + C_1\frac{1}{2}x^2 + C_2 x + C_3, \quad \text{and} \tag{7.24}$$

$$EIw = -\frac{q}{24}x^4 + C_1\frac{1}{6}x^3 + C_2\frac{1}{2}x^2 + C_3 x + C_4. \tag{7.25}$$

Applying the boundary conditions, we have

$$C_1 = qL, \tag{7.26}$$

$$C_2 = -\frac{qL^2}{2}, \quad \text{and} \tag{7.27}$$

$$C_3 = C_4 = 0. \tag{7.28}$$

Then, the deflection is given by

$$EIw(x) = -\frac{q}{24}x^4 + \frac{qL}{6}x^3 - \frac{qL^2}{4}x^2. \tag{7.29}$$

The maximum deflection at $x = L$ is $w_{\max} = -qL^4/(8EI)$.

The deflection is inversely proportional to EI, which is called *flexural rigidity.* While the Young's modulus E represents a material property, the second moment of area I contains the geometrical information. For example (Figure 7.6):

- a rectangular cross-section of width b and height h

$$I = \int_A y^2 dA = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} y^2 dy dz = \frac{bh^3}{12}. \tag{7.30}$$

- a rectangular cross-section of width h and height b

$$I = \int_A y^2 dA = \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} y^2 dy dz = \frac{b^3 h}{12}. \tag{7.31}$$

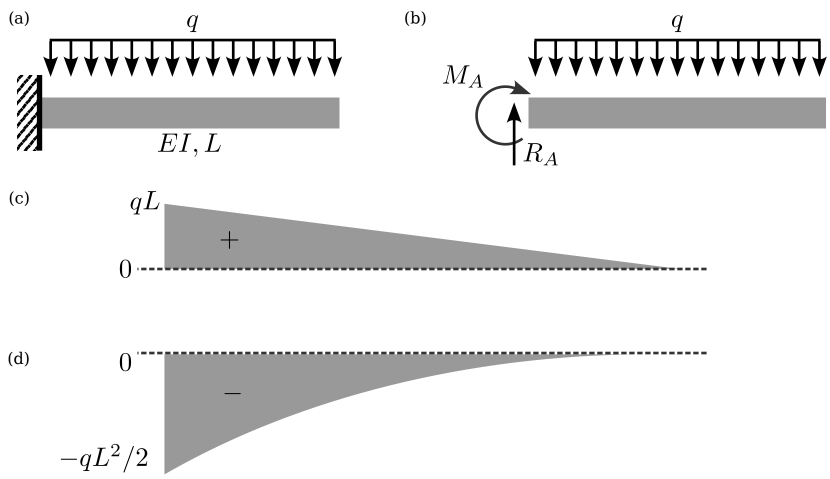


Figure 7.5: Cantilever beam. (a) Structure and load. (b) Freebody diagram. (c) Shear force diagram. (d) Bending moment diagrma.

- an I-beam with the flange thickness t_f and the web height h_w.

$$I = \frac{bh^3}{12} - \frac{(b - t_w)(h - 2t_f)^3}{12}. \tag{7.32}$$

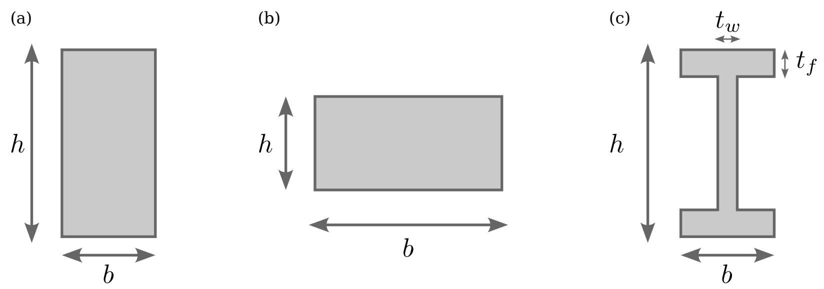


Figure 7.6: Second moments of area for various cross-sections. (a),(b) Rectangualar. (c) I-beam.

7.6 Statically indeterminate structure

Similarly to the previous sections, we exploit the linearity of our model. For example, a two-span continuous beam with three supports (Figure 7.7(c)) is statically indeterminate because there are three unknowns but only two equations.

However, if we are given the mid-point deflections of Figure 7.7(a) and (b), we can use the superposition of the two problems, i.e., apply the compatibility equation. Let the two deflections be denoted by

$$\delta_{(a)} = -5q(2L)^4/(384EI) \quad \text{and} \tag{7.33}$$
$$\delta_{(b)} = -p(2L)^3/(48EI), \tag{7.34}$$

respectively. The compatibility equation then reads

$$\delta_{(a)} + \delta_{(b)} = 0. \tag{7.35}$$

Solving the above equation yields $p = -5qL/4$ (upward), which becomes the reaction force at the mid-support.

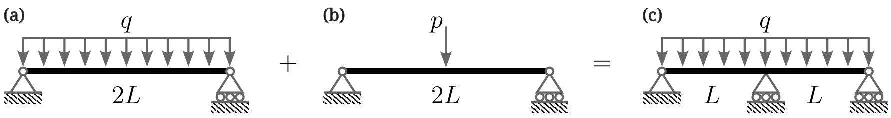


Figure 7.7: Exploiting linearity. (a) A simple beam with a uniform load. (b) With a concentrated load. (c) A beam with three supports that is a linear superposition of the two preceding cases.

The two deflections in (7.34) can be obtained by analyzing the two determinate structures. Thus, we can find the bending moments for each case from the free body diagrams. Then, by integrating twice, we obtain the expressions for the deflections. For each problem, we require two Dirichlet boundary conditions.

From the free body diagrams (Figure 7.8(b) and (c)), we obtain:

$$V_{(a)} = \frac{d}{dx}\left[EI\frac{d^2 w_{(a)}}{dx^2}\right] = -qx + qL, \tag{7.36}$$
$$M_{(a)} = EI\frac{d^2 w_{(a)}}{dx^2} = -\frac{q}{2}x^2 + qLx, \tag{7.37}$$
$$EI\theta_{(a)} = EI\frac{dw_{(a)}}{dx} = -\frac{q}{6}x^3 + \frac{qL}{2}x^2 + C_1, \quad \text{and} \tag{7.38}$$
$$EIw_{(a)} = -\frac{q}{24}x^4 + \frac{qL}{6}x^3 + C_1 x + C_2. \tag{7.39}$$

The integration constants are determined by the boundary conditions $w_{(a)} = 0$ at $x = 0$ and $x = 2L$:

$$C_1 = -\frac{qL^3}{3} \quad \text{and} \quad C_2 = 0. \tag{7.40}$$

Thus, the deflection is

$$EIw_{(a)} = -\frac{q}{24}x^4 + \frac{qL}{6}x^3 - \frac{qL^3}{3}x. \tag{7.41}$$

From the free body diagrams (Figure 7.8(e) and (f)), for $0 < x < L$, we have:

$$V_{(b)} = \frac{d}{dx}\left[EI\frac{d^2w_{(b)}}{dx^2}\right] = \frac{p}{2}, \tag{7.42}$$

$$M_{(b)} = EI\frac{d^2w_{(b)}}{dx^2} = \frac{p}{2}x, \tag{7.43}$$

$$EI\theta_{(b)} = EI\frac{dw_{(b)}}{dx} = \frac{p}{4}x^2 + D_1, \quad \text{and} \tag{7.44}$$

$$EIw_{(b)} = \frac{p}{12}x^3 + D_1x + D_2. \tag{7.45}$$

The integration constants are determined by the conditions $w_{(b)} = 0$ at $x = 0$ and $\theta_{(b)} = 0$ at $x = L$:

$$D_1 = -\frac{pL^2}{4} \quad \text{and} \quad D_2 = 0. \tag{7.46}$$

Here, we use $\theta_{(b)} = 0$ at $x = L$ instead of $w_{(b)} = 0$ at $x = 2L$, considering symmetry about the mid-point. The deflection then reads:

$$EIw_{(b)} = \frac{p}{12}x^3 - \frac{pL^2}{4}x. \tag{7.47}$$

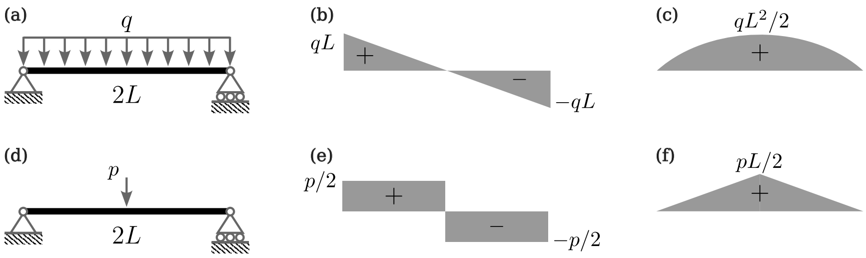


Figure 7.8: Primary structures for Figure 7.7. (a) Structure A and its (b) Shear force and (c) Bending moment diagrams. (d) Structure B and its (e) Shear force and (f) Bending moment diagrams.

The corresponding deflection shapes are shown in Figure 7.9. The deflection of Structure C is the superposition of those of Structures A and B, i.e.,

$$w_{(c)} = w_{(a)} + w_{(b)}\big|_{p=-5qL/4}\,. \tag{7.48}$$

We observe symmetry and smoothness in the deflection shapes. The symmetry is due to the symmetry in geometry and external loading. The smoothness is related to the integrability of the energy functional, which will be discussed later, or, physically, it implies deformation without failure.

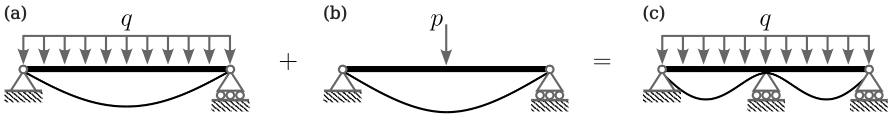


Figure 7.9: Deflection shape for Figure 7.7. (a) Structure A. (b) Structure B. (c) Structure C.

7.7 Shear stress correction

As previously discussed, the assumptions underlying Euler-Bernoulli beam theory neglect shear stress. However, the omitted shear stress can be estimated a posteriori from the freebody diagram shown in Figure 7.10.

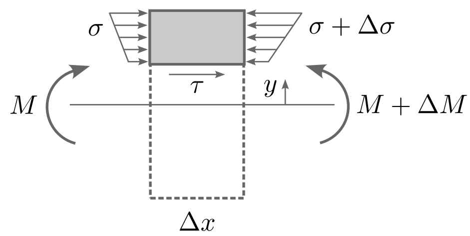


Figure 7.10: Shear stress in an infinitesimal beam element.

Assuming a constant shear stress τ, the equilibrium condition yields

$$0 = \int_{A'} \sigma \, dA + \tau b \Delta x - \int_{A'} (\sigma + \Delta\sigma) \, dA, \tag{7.49}$$

where A' denotes the portion of the cross-sectional area from y to $h/2$, and $b = b(y)$ is the local width of the cross-section. Using the bending stress relation $\sigma = -My/I$, equation (7.49) becomes

$$\begin{aligned} 0 &= -\int_{A'} \frac{My}{I} \, dA + \tau b \Delta x + \int_{A'} \frac{(M + \Delta M)y}{I} \, dA \\ &= \tau b \Delta x + \frac{\Delta M}{I} \int_{A'} y \, dA. \end{aligned} \tag{7.50}$$

Taking the limit as $\Delta x \to 0$, we obtain

$$\begin{aligned} \tau &= \lim_{\Delta x \to 0} \frac{1}{Ib} \frac{\Delta M}{\Delta x} \int_{A'} y \, dA \\ &= \frac{1}{Ib} \frac{dM}{dx} \int_{A'} y \, dA \\ &= \frac{V}{Ib} \int_{A'} y \, dA \\ &= \frac{VQ}{Ib}, \end{aligned} \tag{7.51}$$

where the *first moment of area* Q is defined as

$$Q = \int_{A'} y \, dA. \tag{7.52}$$

Thus, the shear stress τ at a position (x, y) is expressed as

$$\tau(x, y) = \frac{V(x)Q(y)}{I(x)b(y)}, \tag{7.53}$$

where I and V are functions of x.

For a rectangular cross-section of width b and height h, the shear stress becomes (Figure 7.11)

$$\begin{aligned} \tau(y) &= \frac{V}{Ib} \int_{A'} y \, dA \\ &= \frac{V}{I} \int_{y}^{h/2} y' \, dy' \\ &= \frac{V}{I} \left[\frac{1}{2} y'^2 \right]_{y}^{h/2} \\ &= \frac{V}{2I} \left(\frac{h^2}{4} - y^2 \right), \quad -\frac{h}{2} \leq y \leq \frac{h}{2}. \end{aligned} \tag{7.54}$$

Thus, the shear stress varies quadratically with y, vanishing at the top and bottom surfaces.

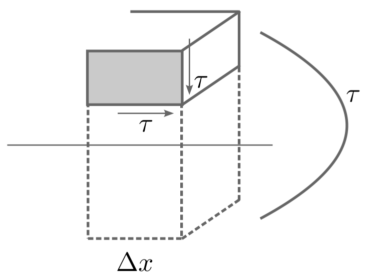


Figure 7.11: Quadratic shear stress in a rectangular cross-section.

Chapter 8

Tensor Transformations

8.1 Transformation of vectors

We consider a rotational coordinate transformation, specifically a transformation from one Cartesian coordinate system to another. Figure 8.1 illustrates various rotational transformations of the form $(x, y, z) \to (x', y', z')$.

Figure 8.1(a) depicts a rotation about the z-axis, for which the transformation is given by

$$\boldsymbol{v}' = \boldsymbol{R}_z \boldsymbol{v} \quad \text{or} \quad \begin{pmatrix} v'_x \\ v'_y \\ v'_z \end{pmatrix} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}. \tag{8.1}$$

Similarly, the transformation matrices for rotations about the y-axis and x-axis are, respectively,

$$\boldsymbol{R}_y = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \quad \text{and} \tag{8.2}$$

$$\boldsymbol{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}. \tag{8.3}$$

Notice that each entry of the rotation matrix is the inner product of the coordinate axes, i.e., $R_{ij} = \boldsymbol{e}_{x'_i} \cdot \boldsymbol{e}_{x_j}$. For example, the coordinate axes corresponding to R_z are

$$\boldsymbol{e}_{x'} = \boldsymbol{e}_x \cos\theta_z + \boldsymbol{e}_y \sin\theta_z, \tag{8.4}$$

$$\boldsymbol{e}_{y'} = \boldsymbol{e}_x (-\sin\theta_z) + \boldsymbol{e}_y \cos\theta_z, \quad \text{and} \tag{8.5}$$

$$\boldsymbol{e}_{z'} = \boldsymbol{e}_z. \tag{8.6}$$

Thus, for a general rotational transformation (Figure 8.1(d)), the rotation matrix reads

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{e}_{x'} \cdot \boldsymbol{e}_x & \boldsymbol{e}_{x'} \cdot \boldsymbol{e}_y & \boldsymbol{e}_{x'} \cdot \boldsymbol{e}_z \\ \boldsymbol{e}_{y'} \cdot \boldsymbol{e}_x & \boldsymbol{e}_{y'} \cdot \boldsymbol{e}_y & \boldsymbol{e}_{y'} \cdot \boldsymbol{e}_z \\ \boldsymbol{e}_{z'} \cdot \boldsymbol{e}_x & \boldsymbol{e}_{z'} \cdot \boldsymbol{e}_y & \boldsymbol{e}_{z'} \cdot \boldsymbol{e}_z \end{bmatrix}. \tag{8.7}$$

Note that such *proper rotation matrices* are *unitary* such that

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I} \tag{8.8}$$

and orientation preserving, i.e.,

$$\det \boldsymbol{R} = 1. \tag{8.9}$$

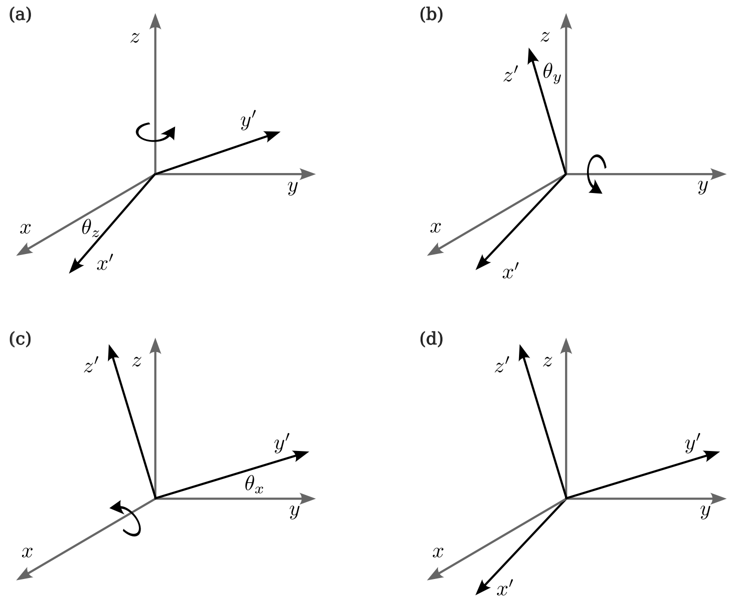


Figure 8.1: Rotational transformation. (a) Rotation about the z axis. (b) Rotation about the y axis. (c) Rotation about the x axis. (d) General rotational transformation.

8.2 Transformation of rank-two tensors

We are now ready to transform rank-two tensors, such as stress and strain tensors. The corresponding transformation rule is derived from the transformation rule for vectors. Consider the transformation of a stress tensor, $\boldsymbol{\sigma} \to \boldsymbol{\sigma}'$. From the definition of the stress tensor, $\boldsymbol{T} = \boldsymbol{\sigma}\boldsymbol{n}$, where $\boldsymbol{T}$ is the traction vector and $\boldsymbol{n}$ is the unit normal vector, we have

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{T} - \boldsymbol{\sigma}\boldsymbol{n} \\ &= \boldsymbol{R}^T\boldsymbol{T}' - \boldsymbol{\sigma}\boldsymbol{R}^T\boldsymbol{n}' \\ &= \boldsymbol{R}\boldsymbol{R}^T\boldsymbol{T}' - \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^T\boldsymbol{n}' \\ &= \boldsymbol{T}' - \boldsymbol{\sigma}'\boldsymbol{n}'. \end{aligned} \tag{8.10}$$

Thus, we have

$$\boldsymbol{\sigma}' = \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^T. \tag{8.11}$$

Strain tensors transform similarly, i.e.,

$$\boldsymbol{\varepsilon}' = \boldsymbol{R}\boldsymbol{\varepsilon}\boldsymbol{R}^T. \tag{8.12}$$

For example, consider a rotation about the z axis. Then, the transformed stress tensor reads

$$\begin{aligned}\boldsymbol{\sigma}' &= \boldsymbol{R}_z\boldsymbol{\sigma}\boldsymbol{R}_z^T \\ &= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ \text{sym.} & & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} & \sigma_{x'z'} \\ & \sigma_{y'y'} & \sigma_{y'z'} \\ \text{sym.} & & \sigma_{z'z'} \end{bmatrix}, \end{aligned} \tag{8.13}$$

where

$$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta, \tag{8.14}$$

$$\sigma_{y'y'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta - \sigma_{xy}\sin 2\theta, \tag{8.15}$$

$$\sigma_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta, \tag{8.16}$$

$$\sigma_{x'z'} = \sigma_{xz}\cos\theta + \sigma_{yz}\sin\theta, \tag{8.17}$$

$$\sigma_{y'z'} = \sigma_{xz}\left(-\sin\theta\right) + \sigma_{yz}\cos\theta, \quad \text{and} \tag{8.18}$$

$$\sigma_{z'z'} = \sigma_{zz}. \tag{8.19}$$

Under certain conditions, stress or strain tensors may be effectively two-dimensional. *Plane stress* refers to the case where $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, while *plane strain* occurs when $u_z = 0$ and $\partial/\partial z = 0$, leading to $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$.

For example, in plane stress, the above transformation can be illustrated graphically, as shown in Figure 8.2.

Some useful identities:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2},$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2},$$
$$\sin 2\theta = 2\sin\theta\cos\theta.$$

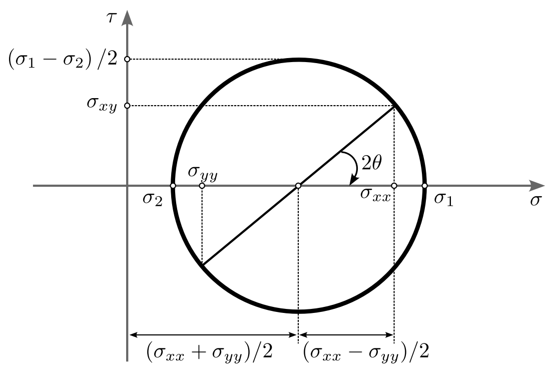


Figure 8.2: Example of Mohr circle when both principal stresses are positive.

The *principal stresses*, σ_1 and σ_2, are the maximum and minimum normal stresses at a given point and occur at a specific orientation θ_p, which can be determined by

$$\begin{aligned} 0 &= \frac{\partial \sigma_{x'x'}}{\partial \theta} \\ &= -\left(\sigma_{xx} - \sigma_{yy}\right)\sin 2\theta_p + 2\sigma_{xy}\cos 2\theta_p, \end{aligned} \tag{8.20}$$

or

$$\tan 2\theta_p = \frac{\sigma_{xy}}{\left(\sigma_{xx} - \sigma_{yy}\right)/2}. \tag{8.21}$$

Then, the principal stresses are

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}. \tag{8.22}$$

There is no shear stress when a certain orientation exhibits principal stresses.

On the other hand, the maximum shear stress appears at

$$\begin{aligned} 0 &= \frac{\partial \sigma_{x'y'}}{\partial \theta} \\ &= -\left(\sigma_{xx} - \sigma_{yy}\right)\cos 2\theta - 2\sigma_{xy}\sin 2\theta \end{aligned} \tag{8.23}$$

or

$$\tan 2\theta_s = -\frac{\left(\sigma_{xx} - \sigma_{yy}\right)/2}{\sigma_{xy}}. \tag{8.24}$$

The corresponding maximum shear stress is

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \frac{\sigma_1 - \sigma_2}{2}. \tag{8.25}$$

Note that the principal stresses are the eigenvalues of the stress tensor, and the corresponding orientation θ_p represents the eigenvectors. The eigenvalues and eigenvectors are defined by

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}, \quad \boldsymbol{x} \neq \boldsymbol{0}. \tag{8.26}$$

The corresponding eigenvalue decomposition of a real symmetric matrix is given by

$$\boldsymbol{A} = \boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{R}^T. \tag{8.27}$$

Here, $\boldsymbol{\Lambda} = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is a diagonal matrix of real-valued eigenvalues and $\boldsymbol{R}$ is an orthogonal (unitary) matrix whose rows are the eigenvectors.

For a general (complex asymmetric) matrix $\boldsymbol{A}$, the eigenvalues are complex-valued and the eigenvectors are not orthogonal. Thus, the decomposition reads

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}.$$

Then, for the plain stress case, we have

$$\begin{bmatrix} \sigma_{xx} - \lambda & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{8.28}$$

A non-trivial solution necessitates

$$\begin{aligned} 0 &= \det \begin{bmatrix} \sigma_{xx} - \lambda & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \lambda \end{bmatrix} \\ &= \left(\sigma_{xx} - \lambda\right)\left(\sigma_{yy} - \lambda\right) - \sigma_{xy}^2 \\ &= \lambda^2 - \left(\sigma_{xx} + \sigma_{yy}\right)\lambda + \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2. \end{aligned} \tag{8.29}$$

The corresponding solution λ is achieved from the quadratic formula:

$$\begin{aligned}\lambda &= \frac{(\sigma_{xx}+\sigma_{yy}) \pm \sqrt{(\sigma_{xx}+\sigma_{yy})^2 - 4\left(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2\right)}}{2} \\ &= \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}, \end{aligned} \tag{8.30}$$

which is identical with (8.22).

Let us revisit the case of pure torsion as an example (Figure 8.3). The stress tensor in Cartesian coordinates is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ \text{sym.} & & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & -\tau & 0 \\ & 0 & 0 \\ \text{sym.} & & 0 \end{bmatrix}. \tag{8.31}$$

Thus, we have a plain stress case. Then, the pricipal stresses are

$$\sigma_{1,2} = \pm\tau, \tag{8.32}$$

where $\theta_p = \pi/4$.

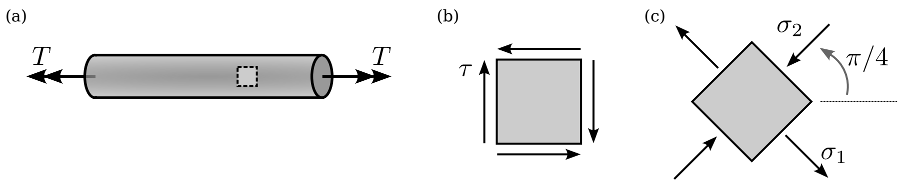


Figure 8.3: Example of stress transformation. (a) Pure torsion. (b) Stress state. (c) Principal stresses.

The corresponding Mohr's circle is Figure 8.4.

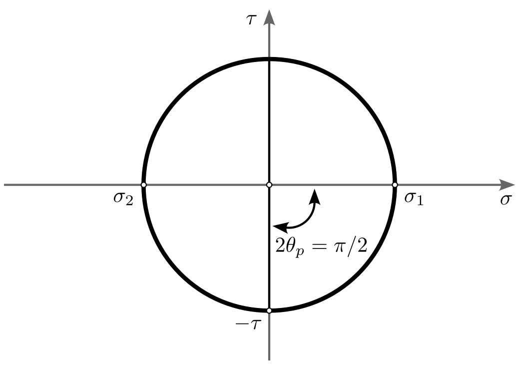


Figure 8.4: Mohr's circle for pure torsion.

8.3 Stress transformation via force equilibrium

In this section, we rederive the stress transformation equations using an alternative approach based on the equilibrium of an infinitesimal element.

We consider a rotational transformation under a plain stress assumption as illustrated in Figure 8.5.

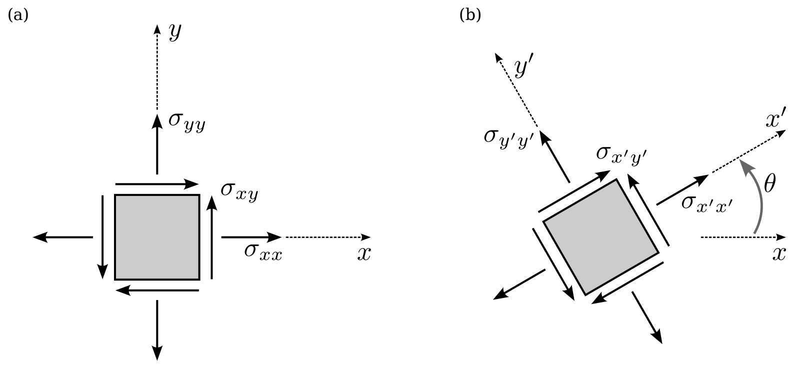


Figure 8.5: Illustration of plain stress tensors in different coordinate systems.

Figure 8.6(a) shows a triangular element with A as the surface area of the inclined side. Here, we apply the force equlibrium equations as

$$\begin{aligned}
0 &= \sum F_{x'} \\
&= \sigma_{x'x'} A - \sigma_{xx} A \cos\theta \cdot \cos\theta - \sigma_{xy} A \cos\theta \cdot \sin\theta \\
&\quad - \sigma_{yy} A \sin\theta \cdot \sin\theta - \sigma_{xy} A \sin\theta \cdot \cos\theta \\
&= \sigma_{x'x'} A - \sigma_{xx} A \frac{1+\cos 2\theta}{2} - \sigma_{xy} A \frac{\sin 2\theta}{2} \\
&\quad - \sigma_{yy} A \frac{1-\cos 2\theta}{2} - \sigma_{xy} A \frac{\sin 2\theta}{2} \\
&= \sigma_{x'x'} - \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2} \cos 2\theta - \sigma_{xy} \sin 2\theta, \qquad (8.33)
\end{aligned}$$

which yields to (8.14), and

$$\begin{aligned}
0 &= \sum F_{y'} \\
&= \sigma_{x'y'} A + \sigma_{xx} A \cos\theta \cdot \sin\theta - \sigma_{xy} A \cos\theta \cdot \cos\theta \\
&\quad - \sigma_{yy} A \sin\theta \cdot \cos\theta + \sigma_{xy} A \sin\theta \cdot \sin\theta \\
&= \sigma_{x'y'} A + \sigma_{xx} A \frac{\sin 2\theta}{2} - \sigma_{xy} A \frac{1+\cos 2\theta}{2} \\
&\quad - \sigma_{yy} A \frac{\sin 2\theta}{2} + \sigma_{xy} A \frac{1-\cos 2\theta}{2} \\
&= \sigma_{x'y'} + \frac{\sigma_{xx}-\sigma_{yy}}{2} \sin 2\theta - \sigma_{xy} \cos 2\theta, \qquad (8.34)
\end{aligned}$$

which yields to (8.16).

Similarly, equlibrium equations on Figure 8.6(b) read

$$\begin{aligned}
0 &= \sum F_{y'} \\
&= \sigma_{y'y'}A - \sigma_{xx}A\sin\theta\cdot\sin\theta + \sigma_{xy}A\sin\theta\cdot\cos\theta \\
&\quad - \sigma_{yy}A\cos\theta\cdot\cos\theta + \sigma_{xy}A\cos\theta\cdot\sin\theta \\
&= \sigma_{y'y'}A - \sigma_{xx}A\frac{1-\cos 2\theta}{2} + \sigma_{xy}A\frac{\sin 2\theta}{2} \\
&\quad - \sigma_{yy}A\frac{1+\cos 2\theta}{2} + \sigma_{xy}A\frac{\sin 2\theta}{2} \\
&= \sigma_{y'y'} - \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta, \qquad (8.35)
\end{aligned}$$

which gives (8.15). The transformation for the shear stress (8.16) is recovered from

$$\begin{aligned}
0 &= \sum F_{x'} \\
&= \sigma_{x'y'}A + \sigma_{xx}A\sin\theta\cdot\cos\theta + \sigma_{xy}A\sin\theta\cdot\sin\theta \\
&\quad - \sigma_{yy}A\cos\theta\cdot\sin\theta - \sigma_{xy}A\cos\theta\cdot\cos\theta \\
&= \sigma_{x'y'}A + \sigma_{xx}A\frac{\sin 2\theta}{2} + \sigma_{xy}A\frac{1-\cos 2\theta}{2} \\
&\quad - \sigma_{yy}A\frac{\sin 2\theta}{2} - \sigma_{xy}A\frac{1+\cos 2\theta}{2} \\
&= \sigma_{x'y'} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta - \sigma_{xy}\cos 2\theta. \qquad (8.36)
\end{aligned}$$

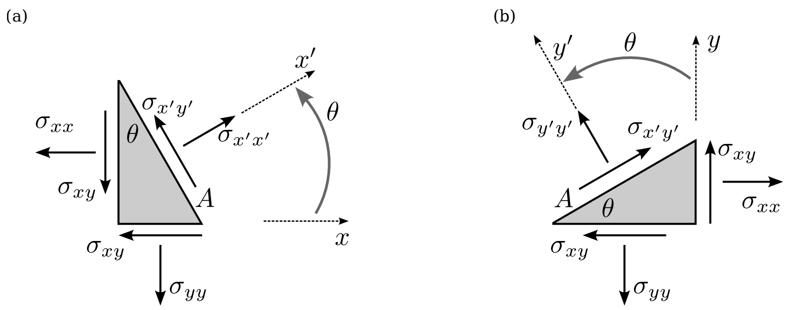


Figure 8.6: Triangular elements (a) for $\sigma_{x'x'}$ and (b) for $\sigma_{y'y'}$.

8.4 Symmetry of elasticity tensor

Recall the stress-strain, or constitutive, relation for linear isotropic elasticity:

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} = 2\mu\boldsymbol{\varepsilon} + \lambda\,(\mathrm{tr}\boldsymbol{\varepsilon})\,\boldsymbol{I}. \qquad (8.37)$$

Below, we apply rotational transformation as

$$
\begin{aligned}
\boldsymbol{\sigma}' &= \boldsymbol{R}\boldsymbol{\sigma}\boldsymbol{R}^T \\
&= \boldsymbol{R}\left(\boldsymbol{C} : \boldsymbol{\varepsilon}\right)\boldsymbol{R}^T \\
&= 2\mu\boldsymbol{R}\boldsymbol{\varepsilon}\boldsymbol{R}^T + \lambda\boldsymbol{R}\left(\mathrm{tr}\boldsymbol{\varepsilon}\right)\boldsymbol{I}\boldsymbol{R}^T \\
&= 2\mu\boldsymbol{\varepsilon}' + \lambda\left(\mathrm{tr}\boldsymbol{\varepsilon}'\right)\boldsymbol{I} \\
&= \boldsymbol{C} : \boldsymbol{\varepsilon}'.
\end{aligned}
\tag{8.38}
$$

Thus, the above relation is invariant under rotational transformations. Moreover, it possesses translational symmetry. Therefore, we have demonstrated the *principle of material frame indifference*, which asserts that a constitutive relation must be *objective*, remaining invariant under a change of observer.

For any proper rotation $\boldsymbol{R}$, we have

$$\mathrm{tr}\left(\boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^T\right) = \mathrm{tr}\boldsymbol{A}.$$

Note that trace is one of the invariants such that $\mathrm{tr}\boldsymbol{A} = \sum_i \lambda_i$, where λ_i are the eigenvalues of the matrix $\boldsymbol{A}$.

Chapter 9

Energy

9.1 Total potential energy

In elasticity, the *total potential energy* is given by

$$\begin{aligned}\Pi &= U - W \\ &= \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Omega - \left(\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\, d\Omega + \int_{\partial\Omega} \boldsymbol{T}\cdot\boldsymbol{u}\, d\Gamma\right).\end{aligned} \tag{9.1}$$

Here, $U = \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Omega$ is the *stored energy*, or *strain energy*, and $W = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, d\Omega + \int_{\partial\Omega} \boldsymbol{T}\cdot\boldsymbol{u}\, d\Gamma$ is the *work done by the external forces*. The *principle of minimum potential energy* states that the displacement fields at equilibrium minimizes the total potential energy among all *admissible functions*.

Thus, the total potential energy is related with the equilibrium equation. For example, consider a spring with the spring constant of k subjected to an external force F_{ext}. The total potential energy of this system is given by

$$\begin{aligned}\Pi &= \int_0^x F d\xi - \int_0^x F_{\text{ext}} d\xi \\ &= \int_0^x k\xi d\xi - F_{\text{ext}} x \\ &= \frac{1}{2}kx^2 - F_{\text{ext}} x.\end{aligned} \tag{9.2}$$

It's minimum appears at $d\Pi/dx = 0$, i.e.,

$$\begin{aligned}0 &= \frac{d\Pi}{dx} \\ &= kx - F_{\text{ext}}.\end{aligned} \tag{9.3}$$

Thus, the first-order necesseary condition gives the equilibrium equation.

In addition, let e an error in solution such that $x = x_{\text{true}} + e$. Plugging it into the energy functional Π, we have

$$\begin{aligned}\Pi\left[x_{\text{true}}+e\right] &= \frac{1}{2}k\left(x_{\text{true}}+e\right)^2 - F\left(x_{\text{true}}+e\right)\\ &= \frac{1}{2}kx_{\text{true}}^2 - Fx_{\text{true}} + \left(kx_{\text{true}}-F\right)e + \frac{1}{2}ke^2\\ &= \Pi_{\text{true}} + \frac{1}{2}ke^2 > \Pi_{\text{true}}. \end{aligned} \tag{9.4}$$

Thus, any function with an error e gives larger total potential energy compared to that of true solution.

9.2 One-dimensional problems

- **Axial load**
  Stress and strain tensors for axial load case are given by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & -\nu\frac{\sigma}{E} & 0 \\ 0 & 0 & -\nu\frac{\sigma}{E} \end{bmatrix}, \tag{9.5}$$

  where $\sigma = Edu/dx$. Then, the corresponding strain energy becomes

$$\begin{aligned} U_{\text{axial}} &= \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Omega \\ &= \frac{1}{2}\int_\Omega \frac{du}{dx}E\frac{du}{dx}d\Omega \\ &= \frac{1}{2}\int_0^L \frac{du}{dx}EA\frac{du}{dx}dx \qquad (9.6)\\ &= \frac{1}{2}\int_0^L \frac{F^2}{EA}dx. \qquad (9.7)\end{aligned}$$

- **Torsion**
  For torsion, we have (in cylindrical coordinate $\{x, \rho, \theta\}$)

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & G\rho\frac{d\varphi}{dx} \\ 0 & G\rho\frac{d\varphi}{dx} & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\rho\frac{d\varphi}{dx} \\ 0 & \frac{1}{2}\rho\frac{d\varphi}{dx} & 0 \end{bmatrix}. \tag{9.8}$$

  Then, the strain energy reads

$$\begin{aligned} U_{\text{torsion}} &= \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Omega \\ &= \frac{1}{2}\int_\Omega G\rho\frac{d\varphi}{dx}\rho\frac{d\varphi}{dx}d\Omega \\ &= \frac{1}{2}\int_0^L \frac{d\varphi}{dx}G\left(\int_A \rho^2 dA\right)\frac{d\varphi}{dx}dx \\ &= \frac{1}{2}\int_0^L \frac{d\varphi}{dx}GJ\frac{d\varphi}{dx}dx \qquad (9.9)\\ &= \frac{1}{2}\int_0^L \frac{T^2}{GJ}dx. \qquad (9.10)\end{aligned}$$

- **Bending**
  We neglect Poisson effect for bending, which gives

$$\boldsymbol{\sigma} = \begin{bmatrix} -E\frac{d^2w}{dx^2}y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} -\frac{d^2w}{dx^2}y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9.11}$$

  Then,

$$\begin{aligned} U_{\text{bending}} &= \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Omega \\ &= \frac{1}{2}\int_\Omega \left(-\frac{d^2w}{dx^2}y\right)\left(-E\frac{d^2w}{dx^2}y\right) d\Omega \\ &= \frac{1}{2}\int_0^L \frac{d^2w}{dx^2}E\left(\int_A y^2 dA\right)\frac{d^2w}{dx^2}dx \\ &= \frac{1}{2}\int_0^L \frac{d^2w}{dx^2}EI\frac{d^2w}{dx^2}dx \qquad (9.12) \\ &= \frac{1}{2}\int_0^L \frac{M^2}{EI}dx. \qquad (9.13) \end{aligned}$$

- **Bending shear**
  The posteriori shear estimation from bending gives

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \frac{VQ}{Ib} & 0 \\ \frac{VQ}{Ib} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} 0 & \frac{1}{2G}\frac{VQ}{Ib} & 0 \\ \frac{1}{2G}\frac{VQ}{Ib} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9.14}$$

  Then,

$$\begin{aligned} U_{\text{bending shear}} &= \frac{1}{2}\int_\Omega \overline{\varepsilon} : \sigma d\Omega \\ &= \frac{1}{2}\int_\Omega \left(\frac{1}{G}\frac{VQ}{Ib}\right)\left(\frac{VQ}{Ib}\right) d\Omega \\ &= \frac{1}{2}\int_0^L \frac{V^2}{G}\left(\int_A \frac{Q^2}{I^2b^2}dA\right) dx \\ &= \frac{1}{2}\int_0^L f_s\frac{V^2}{GA}dx. \qquad (9.15) \end{aligned}$$

  Here, the *form factor* f_s is defined as

$$f_s(x) = \frac{A(x)}{I^2(x)}\int_A \frac{Q^2(x,y)}{b^2(y)}dA, \tag{9.16}$$

  where for a rectangle $f_s = 6/5$, for a circle $f_s = 10/9$, and for a thin-walled tube $f_s = 2$.

9.3 Functional derivative

Differentiation of a function $f(x) : \mathbb{R} \to \mathbb{R}$ with respect to its argument $x \in \mathbb{R}$ is defined as

$$\frac{df}{dx} = \lim_{\epsilon \to 0}\frac{f(x+\epsilon) - f(x)}{\epsilon}. \tag{9.17}$$

As an example, the derivative of a polynomial function reads

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{\epsilon\to 0}\frac{(x+\epsilon)^n - x^n}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{x^n + n\epsilon x^{n-1} + \sum_{k=2}^{n}\begin{pmatrix} n\\ k\end{pmatrix} x^{n-k}\epsilon^k - x^n}{\epsilon}\\ &= \lim_{\epsilon\to 0}\left(nx^{n-1} + \mathcal{O}(\epsilon)\right)\\ &= nx^{n-1}.\end{aligned} \tag{9.18}$$

A direction $\boldsymbol{v}$ must be introduced to differentiate a multivariate function $f(\boldsymbol{x}) : \mathbb{R}^N \to \mathbb{R}$, i.e.,

$$D_{\boldsymbol{v}} f(\boldsymbol{x}) = \lim_{\epsilon\to 0}\frac{f(\boldsymbol{x}+\epsilon\boldsymbol{v}) - f(\boldsymbol{x})}{\epsilon} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} f(\boldsymbol{x}+\epsilon\boldsymbol{v}). \tag{9.19}$$

As an example, let $f(x,y) = 2x + y$ and $\boldsymbol{v} = (1,1)$; then,

$$\begin{aligned}D_{\boldsymbol{v}} f(\boldsymbol{x}) &= \lim_{\epsilon\to 0}\frac{2(x+\epsilon) + (y+\epsilon) - 2x - y}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{2\epsilon+\epsilon}{\epsilon}\\ &= 3.\end{aligned} \tag{9.20}$$

If $D_{\boldsymbol{v}} f$ exists for any direction $\boldsymbol{v}$ and $\boldsymbol{v} \to D_{\boldsymbol{v}} f$ is linear, then, there exits a vector $\boldsymbol{g}$ such that

$$D_{\boldsymbol{v}} f = \boldsymbol{g}\cdot\boldsymbol{v}. \tag{9.21}$$

The vector $\boldsymbol{g}$ is identified as the gradient of f. Revisiting the above example,

$$\begin{aligned}D_{\boldsymbol{v}} f(\boldsymbol{x}) &= \lim_{\epsilon\to 0}\frac{2(x+\epsilon v_x) + (y+\epsilon v_y) - 2x - y}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{\epsilon(2,1)\cdot(v_x, v_y)}{\epsilon}\\ &= (2,1)\cdot\boldsymbol{v}.\end{aligned} \tag{9.22}$$

Then we have the gradient $\boldsymbol{g} = (2,1)$.

Functional derivatives follows the same rule as discussed above. For example, consider the following energy functional $\Pi[u] : \mathcal{V} \to \mathbb{R}$

$$\Pi[u] = \frac{1}{2}\int_0^L \frac{du}{dx} EA\frac{du}{dx}dx - \int_0^L fu dx. \tag{9.23}$$

Here, $\mathcal{V}$ is a function space of admissible functions. Its directional derivative, or *Gateaux derivative*, in the direction of $v \in \mathcal{W}$ reads

$$
\begin{aligned}
D_v\Pi[u] &= \lim_{\epsilon\to 0} \frac{\Pi[u+\epsilon v] - \Pi[u]}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\frac{1}{2}\int_0^L \frac{d(u+\epsilon v)}{dx} EA \frac{d(u+\epsilon v)}{dx} dx - \int_0^L f(u+\epsilon v) dx \right. \\
&\quad \left. - \left(\frac{1}{2}\int_0^L \frac{du}{dx} EA \frac{du}{dx} dx - \int_0^L f u dx\right)\right] \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\frac{1}{2}\int_0^L \frac{du}{dx} EA \frac{du}{dx} dx + \int_0^L \frac{du}{dx} EA \frac{d\epsilon v}{dx} dx + \frac{1}{2}\int_0^L \frac{d\epsilon v}{dx} EA \frac{d\epsilon v}{dx} dx \right. \\
&\quad \left. - \int_0^L f u dx - \int_0^L f \epsilon v dx - \frac{1}{2}\int_0^L \frac{du}{dx} EA \frac{du}{dx} dx + \int_0^L f u dx \right] \\
&= \int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx - \int_0^L f v dx. \qquad (9.24)
\end{aligned}
$$

If we wish to find the gradient g, we need to rearrange the above directional derivative such that v is linearly separated, i.e.,

$$
D_v\Pi[u] = (g, v) = \int_0^L g v dx. \qquad (9.25)
$$

We can achieve the above by integration by parts, i.e.,

$$
\begin{aligned}
D_v\Pi[u] &= \int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx - \int_0^L f v dx \\
&= \left[v EA \frac{du}{dx}\right]_0^L - \int_0^L v \frac{d}{dx}\left[EA\frac{du}{dx}\right] - \int_0^L v f dx \\
&= \int_0^L v\left(\frac{d}{dx}\left[EA\frac{du}{dx}\right] + f\right) dx. \qquad (9.26)
\end{aligned}
$$

In the above, we assumed that the boundary term vanishes from the choice of $\mathcal{V}$ and $\mathcal{W}$. Then, we have the gradient as

$$
g = \frac{d}{dx}\left[EA\frac{du}{dx}\right] + f. \qquad (9.27)
$$

The principle of minimum potential energy requires that $D_v\Pi[u] = 0$ for any v. Consequently, we require $g = 0$, which gives the governing equation:

$$
\frac{d}{dx}\left[EA\frac{du}{dx}\right] + f = 0. \qquad (9.28)
$$

9.4 Applications

Define a trial solution u with an error e, such that $u = u_{\text{true}} + e$. Then the total potential energy for axial load reads

$$\begin{aligned}\Pi\left[u_{\text{true}}+e\right] =& \frac{1}{2}\int_0^L \frac{d\left(u_{\text{true}}+e\right)}{dx} EA \frac{d\left(u_{\text{true}}+e\right)}{dx} dx - \int_0^L f\left(u_{\text{true}}+e\right) dx \\ =& \frac{1}{2}\int_0^L \frac{du_{\text{true}}}{dx} EA \frac{du_{\text{true}}}{dx} dx - \int_0^L u_{\text{true}} f dx \\ &+ \int_0^L \frac{du_{\text{true}}}{dx} EA \frac{de}{dx} dx - \int_0^L e f dx + \frac{1}{2}\int_0^L \frac{de}{dx} EA \frac{de}{dx} dx \\ =& \Pi\left[u_{\text{true}}\right] + \frac{1}{2}\int_0^L \frac{de}{dx} EA \frac{de}{dx} dx \geq \Pi\left[u_{\text{true}}\right]. \end{aligned} \tag{9.29}$$

Thus, any error e always overestimates the total potential energy.

9.5 Admissible functions

Not all functions are allowed as candidates of a given problem. Instead, a trial function is admissible only if it satisfies two requirements:

1. Dirichlet boundary condition *and*
2. Finite strain energy.

Dirichlet data prescribe kinematic quantities—displacement for axial loading and torsion, or deflection and rotation (slope) for beams. Although meeting the corresponding Neumann (force or moment) boundary conditions is desirable, it is not necessary.

Finite strain energy is ensured when the relevant strain measure is integrable:

$$\int_0^L \frac{du}{dx} EA \frac{du}{dx} dx < \infty \quad \text{(axial load)}, \tag{9.30}$$

$$\int_0^L \frac{d\varphi}{dx} GJ \frac{d\varphi}{dx} dx < \infty \quad \text{(torsion)}, \quad \text{and} \tag{9.31}$$

$$\int_0^L \frac{d^2w}{dx^2} EI \frac{d^2w}{dx^2} dx < \infty \quad \text{(bending)}. \tag{9.32}$$

For example, consider an axially loaded bar fixed at one end and free at the other end, i.e., $u = 0$ at $x = 0$ and $EAdu/dx = 0$ at $x = L$. The corresponding function space $\mathcal{U}$ of admissible functions reads

$$\mathcal{U} = \left\{ u \;:\; u(0) = 0,\; \int_0^L \left(\frac{du}{dx}\right)^2 dx < \infty,\; \int_0^L u^2 dx < \infty \right\}. \tag{9.33}$$

Thus, a function with a finite jump such as

$$u\left(x\right) = \begin{cases} x & 0 \leq x < L/2 \\ L & L/2 \leq x \leq L \end{cases} \tag{9.34}$$

is not admissible because the its derivative at the jump is described by a Dirac delta function, which is not square-integrable.

The extra integrability condition on u ensures finite norm of the function, which is required for existence and uniqueness of a solution. A function space with the two integrability conditions is called *Sobolev space* and reads

$$H^1(0,L) = \left\{ u \;:\; \int_0^L \left(\frac{du}{dx}\right)^2 dx < \infty,\; \int_0^L u^2 dx < \infty \right\}. \tag{9.35}$$

Thus, the function space (9.33) can be equivalently stated as

$$\mathcal{U} = \left\{ u \in H^1(0,L) \;:\; u(0) = 0 \right\}. \tag{9.36}$$

On the other hand, the admissible function space for a simple beam ($u(0) = u(L) = 0$ and $M(0) = M(L) = 0$) reads

$$\mathcal{U} = \left\{ w \in H^2(0,L) \;:\; w(0) = w(L) = 0 \right\}, \tag{9.37}$$

where

$$H^2(0,L) = \left\{ u \;:\; \int_0^L \left(\frac{d^2u}{dx^2}\right)^2 dx < \infty,\; \int_0^L \left(\frac{du}{dx}\right)^2 dx < \infty,\; \int_0^L u^2 dx < \infty \right\}. \tag{9.38}$$

Here, each integrability condition corresponds to finite energy, finite slope, and finite deflection, respectively. Thus, an admissible function must be differentiable so that its second-order derivative is continuous.

Chapter 10

Stability (buckling)

10.1 Buckling of a discrete system

A structure may fail when its material fractures due to large strain or stress. In this chapter, we discuss a different type of failure called *buckling*, which involves a loss of system-level stiffness.

We start our discussion with a system of two rigid bars connected by a rotational spring (Figure 10.1(a)). The rotational spring resists "bending" of the two-bar system due to the axial load p via a reaction moment proportional to the angle θ such that

$$M = 2\theta k. \tag{10.1}$$

Here, k is the rotational stiffness. For the structure to be stable, the reaction moment must be balanced by the moment due to the load p, i.e.,

$$\begin{aligned} 0 &= M - L\theta p \\ &= 2\theta k - \frac{L}{2}\theta p \\ &= \left(2k - \frac{L}{2}p\right)\theta. \end{aligned} \tag{10.2}$$

In the above, we observe that the equation is satisfied regardless of the value of θ when $2k - Lp = 0$. Such load is called the *critical load* and reads

$$p_{\mathrm{cr}} = \frac{4k}{L}. \tag{10.3}$$

Thus, any values of θ, including zero, are solutions, i.e, the system is *unstable*. Note that the critical load is inversely proportional to the length L.

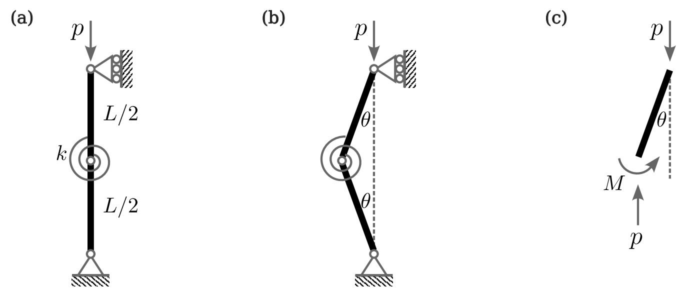


Figure 10.1: Two rigid bars connected by a rotational spring. (a) System. (b) Deformed shape. (c) Freebody diagram.

10.2 Buckling of a column

Let's consider a continuum counterpart of the previous example, i.e, a column subject to an axial load (Figure 10.2). Let M denote the moment due to bending, we construct moment equilibrium equation such that

$$\begin{aligned} 0 &= M + pw \\ &= EI\frac{d^2w}{dx^2} + pw \\ &= \left(EI\frac{d^2}{dx^2} + p\right)w. \end{aligned} \tag{10.4}$$

Again, we seek for a critical load p such that the above equation is satisfies regardless of the magnitude of w.

We try $w = Ae^{ikx} + Be^{-ikx}$ as a trial solution. The trial solution satisfies the governing equation when $k^2 = p/EI$. Then, the boundary condition

$$w(0) = w(L) = 0 \tag{10.5}$$

gives

$$A + B = 0 \quad \text{and} \quad Ae^{ikL} + Be^{-ikL} = 0. \tag{10.6}$$

We can write the above boundary conditions in a matrix form as

$$\begin{bmatrix} 1 & 1 \\ e^{ikL} & e^{-ikL} \end{bmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.7}$$

The solution, except for the trivial solution $A = B = 0$, exists only if the matrix is singular, i.e.,

$$\begin{aligned} 0 &= \det \begin{bmatrix} 1 & 1 \\ e^{ikL} & e^{-ikL} \end{bmatrix} \\ &= e^{-ikL} - e^{ikL} \\ &= -2\sin kL. \end{aligned} \tag{10.8}$$

Thus, we have

$$kL = n\pi \tag{10.9}$$

or, since $k = \sqrt{p/EI}$,

$$p = \frac{n^2\pi^2 EI}{L^2} \equiv p_{\mathrm{cr}}, \quad n = 1, 2, \ldots. \tag{10.10}$$

The corresponding deflection w is

$$w(x) = A \sin \frac{n\pi}{L} x, \tag{10.11}$$

where the magnitude A is not determined. For each n, we have a critical load and corresponding buckling shape. Note also that different boundary conditions lead to differnt critical loads and buckling shapes.

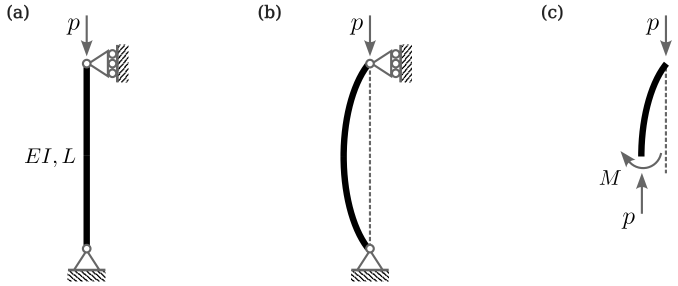


Figure 10.2: A column subject to an axial load. (a) System. (b) Deformed shape. (c) Freebody diagram.

The modified beam equation considering $p - \Delta$ *effect* reads

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] + p\frac{d^2w}{dx^2} = q. \tag{10.12}$$

The corresponding total potential energy reads

$$\Pi[w] = \frac{1}{2}\int_0^L \left(\frac{d^2w}{dx^2}EI\frac{d^2w}{dx^2} - p\frac{dw}{dx}\frac{dw}{dx}\right)dx - \int_0^L qwdx. \tag{10.13}$$

Then, computing the total potential energy for the problem above, we have

$$\begin{aligned}\Pi[w] &= \frac{1}{2}\left(\frac{n^2\pi^2}{L^2}\right)^2 A^2 EI \int_0^L \sin^2\frac{n\pi x}{L}dx - \frac{p}{2}\left(\frac{n^2\pi^2}{L^2}\right)A^2\int_0^L \cos^2\frac{n\pi x}{L}dx \\ &= \frac{A^2}{2}\frac{L}{2}\frac{n^2\pi^2}{L^2}\left(\frac{n^2\pi^2 EI}{L^2} - p\right). \end{aligned}\tag{10.14}$$

The above potential energy vanishes when $p = p_{\mathrm{cr}}$ regardless of A, which is the case when the principle of minimum potential energy does not hold.

10.3 Effect of Boundary Conditions

As mentioned earlier, the stability of a column is highly sensitive to its boundary conditions.

Consider first a column with both ends fixed, as shown in Figure 10.3. The boundary conditions are:

$$w(0) = w(L) = 0 \quad \text{and} \tag{10.15}$$

$$\frac{dw(0)}{dx} = \frac{dw(L)}{dx} = 0. \tag{10.16}$$

We retain the moment reaction M_o as an unknown and formulate the equilibrium equation based on the freebody diagram in Figure 10.3(c), which reads

$$\begin{aligned} 0 &= M + pw - M_o \\ &= EI\frac{d^2w}{dx^2} + pw - M_o. \end{aligned} \tag{10.17}$$

We assume a solution of the form

$$w = Ae^{ikx} + Be^{-ikx} + \frac{M_o}{p}, \tag{10.18}$$

where $k^2 = p/(EI)$. The given choice of solution form satisfies the equlibrium equation for arbitrary A, B and M_o. Then, we apply the boundary conditions, which gives

$$\begin{cases} A + B + \dfrac{M_o}{p} = 0 \\ Ae^{ikL} + Be^{-ikL} + \dfrac{M_o}{p} = 0 \\ A(ik) + B(-ik) = 0 \\ A(ik)e^{ikL} + B(-ik)e^{-ikL} = 0 \end{cases}. \tag{10.19}$$

Solving the first and thrid equations gives $A = B = -M_o/(2p)$. Then,

$$\begin{cases} -\dfrac{M_o}{p}\cos kL + \dfrac{M_o}{p} = 0 \\ \dfrac{M_o}{p}k\sin kL = 0 \end{cases} \Rightarrow \cos kL = 1 \text{ and } \sin kL = 0. \tag{10.20}$$

For nontrivial solutions, the following condition must hold

$$p = \frac{(2\pi)^2 n^2 EI}{L^2}. \tag{10.21}$$

The corresponding deflection shape becomes

$$w = \frac{M_o}{p}\left(1 - \cos\frac{2n\pi x}{L}\right). \tag{10.22}$$

Next, conider a case with one fixed support and one free support (Figure 10.4). Here, we denote a small deflection at the end ($x = L$) as δ. The corresponding boundary conditions are

$$w(0) = \frac{dw(0)}{dx} = 0. \quad \text{and} \tag{10.23}$$

$$w(L) = \delta. \tag{10.24}$$

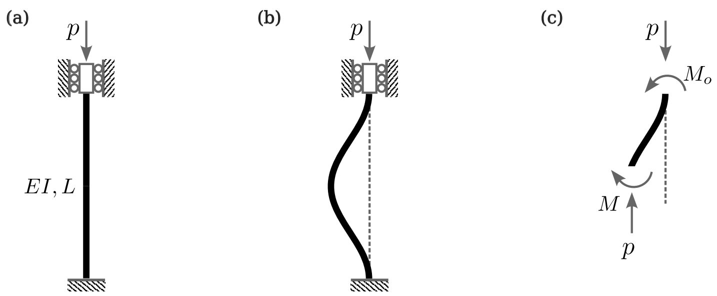


Figure 10.3: A column with both ends fixed subject to an axial load. (a) System. (b) Deformed shape. (c) Freebody diagram.

The equilibrium equation reads

$$\begin{aligned} 0 &= M + pw - p\delta \\ &= EI\frac{d^2w}{dx^2} + pw - p\delta. \end{aligned} \tag{10.25}$$

The ansatz is given by ($k^2 = p/EI$)

$$w(x) = Ae^{ikx} + Be^{-ikx} + \delta. \tag{10.26}$$

Applying the boundary conditions, we have

$$\begin{cases} A + B + \delta = 0 \\ A(ik) + B(-ik) = 0 \\ Ae^{ikL} + Be^{-ikL} + \delta = \delta \end{cases} \tag{10.27}$$

or

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ e^{ikL} & e^{-ikL} & 0 \end{bmatrix} \begin{pmatrix} A \\ B \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10.28}$$

For the existence of nontrivial solution, we require

$$\cos kL = 0 \Rightarrow kL = \frac{(2n-1)\pi}{2}. \tag{10.29}$$

Thus, we have

$$p = \frac{(2n-1)^2 \pi^2 EI}{4L^2}. \tag{10.30}$$

The corresponding deflection shape is

$$w(x) = \delta\left[1 - \cos\frac{(2n-1)\pi x}{2L}\right]. \tag{10.31}$$

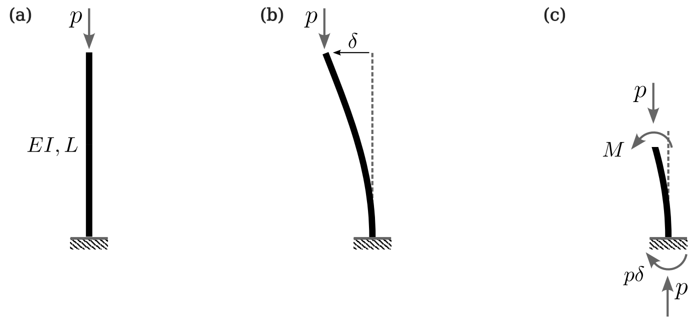


Figure 10.4: A column with fixed and free supports subject to an axial load. (a) System. (b) Deformed shape. (c) Freebody diagram.

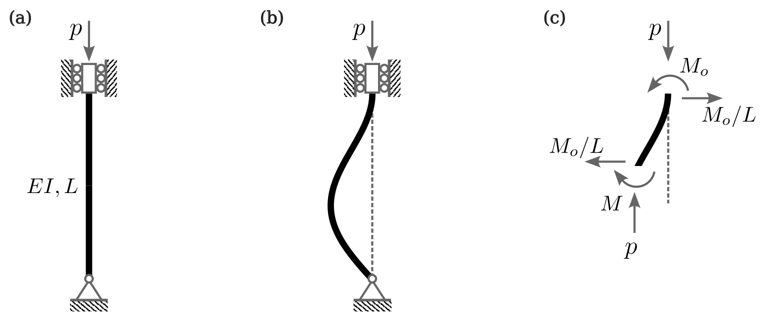


Figure 10.5: A column with hinged and fixed supports subject to an axial load. (a) System. (b) Deformed shape. (c) Freebody diagram.

Figure 10.5 shows a column with one hinged support and one fixed support such that

$$w(0) = 0 \quad \text{and} \tag{10.32}$$

$$w(L) = \frac{dw(L)}{dx} = 0. \tag{10.33}$$

The corresponding equilibrium equation reads

$$\begin{aligned} 0 &= M + pw - M_o + (L - x)\frac{M_o}{L} \\ &= EI\frac{d^2w}{dx^2} + pw - \frac{M_o x}{L}. \end{aligned} \tag{10.34}$$

We take

$$w = Ae^{ikx} + Be^{-ikx} + \frac{M_o}{p}\frac{x}{L}. \tag{10.35}$$

Applying the boundary conditions we have

$$\begin{cases} A + B = 0 \\ Ae^{ikL} + Be^{-ikL} + \dfrac{M_o}{p} = 0 \\ A(ik)e^{ikL} + B(-ik)e^{-ikL} + \dfrac{M_o}{pL} = 0 \end{cases} \tag{10.36}$$

or

$$\begin{bmatrix} 1 & 1 & 0 \\ e^{ikL} & e^{-ikL} & \dfrac{1}{p} \\ (ik)e^{ikL} & (-ik)e^{-ikL} & \dfrac{1}{pL} \end{bmatrix} \begin{pmatrix} A \\ B \\ M_o \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{10.37}$$

A nontrivial solution exists if the determinant of the matrix vanishes, which gives

$$\tan kL = kL. \tag{10.38}$$

Then, the smallest nontrivial solution occurs approximately at $kL \approx 4.49$ as shown in Figure 10.6. Thus, the critical load is

$$p_{cr} = \frac{20.2EI}{L^2} \tag{10.39}$$

and the corresponding deflected shape reads

$$w = \frac{M_o}{p}\left[\frac{x}{L} + 1.02\sin\left(4.49\frac{x}{L}\right)\right]. \tag{10.40}$$

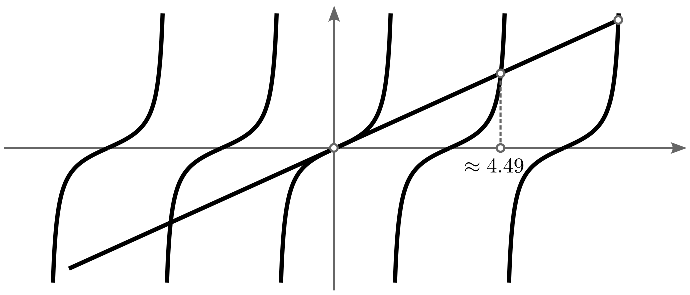


Figure 10.6: Solution set for $\tan kL = kL$.

Bibliography

[Crandall et al., 2012] Crandall, S. H., Dahl, N. C., Lardner, T. J., and Sivakumar, M. S. (2012). *An Introduction to Mechanics of Solids: (In SI Units)*. Tata McGraw Hill Education Private Limited, New Delhi, 3 edition.

[Gel'Fand and Shilov, 1964] Gel'Fand, I. M. and Shilov, G. E. (1964). *Generalized Functions: Properties and Operations*, volume 1. Academic Press, New York and London.

[Goodno and Gere, 2017] Goodno, B. J. and Gere, J. M. (2017). *Mechanics of Materials*. Cengage Learning, Boston, MA, 9 edition.

[Hibbeler, 2010] Hibbeler, R. C. (2010). *Mechanics of Materials*. Prentice Hall, Upper Saddle River, NJ, 8 edition.

[Lee, 2022a] Lee, H. S. (2022a). Lecture notes for structural analaysis 1. Seoul National University.

[Lee, 2022b] Lee, H. S. (2022b). Lecture notes for structural analaysis 2. Seoul National University.