

# Midterm Exam 1 - Mechanics of Materials

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**Problem 1. (History-dependent constitutive relation)** Suppose we wish to distinguish a raw egg from a hard-boiled egg. The test consists of spinning the egg on a flat surface and briefly stopping it by pressing a fingertip against the shell. A hard-boiled egg, behaving as a rigid body, remains at rest once released. A raw egg, however, resumes spinning: its internal fluid retains angular momentum during the brief contact and subsequently re-engages the shell through viscous coupling. Such behavior can be modelled by a history-dependent inertia, i.e.,

$$\tau(t) = I_0 \ddot{\theta}(t) + \int_{-\infty}^{\infty} K(t-s) \ddot{\theta}(s) ds, \quad (1)$$

where

$$K(t) = (I_{\infty} - I_0) \frac{e^{-t/t^*}}{t^*} H(t), \quad H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (2)$$

Here,  $\tau$  is the applied torque,  $\theta$  is the rotation angle,  $t^* > 0$  is the characteristic relaxation time, and  $I_0 < I_{\infty}$ .

- Explain the physical meaning of  $t^*$ ,  $I_{\infty}$ , and  $I_0$ .
- Discuss how the kernel  $K$  reflects the memory effect in the rotational motion of a raw egg.
- For a hard-boiled egg, we expect to recover rigid-body rotation, i.e.,

$$\tau = I_{\infty} \ddot{\theta}. \quad (3)$$

Show that (1) reduces to (3) in the appropriate limiting case.

*Hint.* Use the following limits if needed:

$$\lim_{t^* \rightarrow \infty} \frac{e^{-t/t^*}}{t^*} H(t) = 0, \quad (4)$$

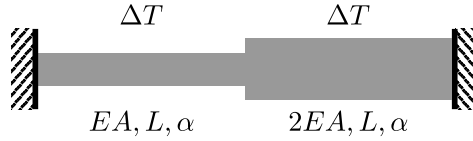
$$\lim_{t^* \rightarrow 0^+} \frac{e^{-t/t^*}}{t^*} H(t) = \delta(t), \quad (5)$$

where  $\delta(t)$  is the Dirac delta, defined by the sifting property

$$\int_{-\infty}^{\infty} \delta(t-s) f(s) ds = f(t). \quad (6)$$

**Problem 2. (Thermal expansion)** A stepped bar is fixed at both ends and consists of two segments of equal length  $L$ . Segment 1 has cross-sectional area  $A_1 = A$  and segment 2 has cross-sectional area  $A_2 = 2A$ , where  $A$  is a reference area. Both segments are made of the same linear elastic material with Young's modulus  $E$  and coefficient of thermal expansion  $\alpha$ . The entire bar is subjected to a uniform temperature increase  $\Delta T$ . The thermal strain for a uniform temperature increase is  $\varepsilon_T = \alpha\Delta T$ .

- State the equilibrium equation and the compatibility equation.
- Using the equations from part (1), determine the reaction force at each fixed end.
- Draw the axial force diagram.
- Determine the axial stress in each segment.



Problem 2.

**Problem 3. (Hydrostatic pressure)** A linear elastic, isotropic body with Young's modulus  $E$  and Poisson's ratio  $\nu$  is subjected to a uniform hydrostatic pressure  $p > 0$ , so that the stress tensor takes the form

$$\boldsymbol{\sigma} = -p\mathbf{I}, \quad \text{or, in components,} \quad \sigma_{ij} = -p\delta_{ij}. \quad (7)$$

In matrix form:

$$\boldsymbol{\sigma} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}. \quad (8)$$

- Determine all components of the strain tensor  $\boldsymbol{\varepsilon}$  using the generalized Hooke's law:

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\text{tr}(\boldsymbol{\sigma})\mathbf{I}, \quad \text{or, in components,} \quad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}. \quad (9)$$

- Compute the dilatation  $e := \text{tr}(\boldsymbol{\varepsilon}) = \varepsilon_{kk}$  by substituting the result of part (a). Express  $e$  in terms of  $p$ ,  $E$ , and  $\nu$ .
- Consider a unit cube with undeformed side lengths 1. After deformation, each side length changes according to the normal strains found in part (a). Write the exact expression for the deformed volume  $V'$ , then retain only terms linear in the strains:

$$V' = V'_{\text{lin}} + \mathcal{O}(e^2). \quad (10)$$

- The volumetric strain is defined as

$$\frac{\Delta V}{V} = \frac{V'_{\text{lin}} - V}{V}. \quad (11)$$

Show that this equals  $e = \text{tr}(\boldsymbol{\varepsilon})$  obtained in part (b), and identify the bulk modulus  $K$  from the relation

$$-p = K e, \quad (12)$$

expressing  $K$  in terms of  $E$  and  $\nu$ . What does the limit  $\nu \rightarrow \frac{1}{2}$  imply physically?