Midterm Exam 1 - Mechanics of Materials

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Problem 1. (Stress tensor) Consider a two-dimensional bar problem subjected to a uniform normal stress (Figure 1(a)). Let $\boldsymbol{T}$ denote the traction and a denote the cross-sectional length at the boundaries of the bar; then, the force $\boldsymbol{F}$ on the boundaries share the same magnitude with opposite directions to satisfy the balance of forces (or the equilibrium equation), where

$$\boldsymbol{F} = \boldsymbol{T}a. \tag{1}$$

The components of the traction are given by

$$\boldsymbol{T} = \begin{pmatrix} F/a \\ 0 \end{pmatrix}, \tag{2}$$

where $F = |\boldsymbol{F}|$.

We seek to determine the components of the stress tensor $\boldsymbol{\sigma}$ such that

$$\boldsymbol{T} = \boldsymbol{\sigma n} \quad \text{or} \quad \begin{pmatrix} T_x \\ T_y \end{pmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix}, \tag{3}$$

where $\boldsymbol{n}$ is the unit outward normal vector on a surface. Figure 1(b) and (c) are freebody diagrams of different cross-sections, where

$$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{n}_2 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{4}$$

1. Using the balance of forces, find the components of $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ in terms of F, a and θ. (Note that (1)-(1)' and (2)-(2)' have different cross-sectional lengths.)
2. Then, find the components of stress tensor from the above result and the definition of the stress tensor (3).

Problem 2 (Hooke's law and Poisson's ratio) We assume Hooke's law as the relation between the normal stress and normal strain, i.e.,

$$\sigma = E\varepsilon. \tag{5}$$

Here, σ is the normal stress, E is the Young's modulus, and ε is the normal strain. In the above case (Figure 1(a)), the normal (horizontal) stress is $\sigma = F/a$.

1. Let b the horizontal length of the bar, find the normal strain ε and the horizontal elongation δ.
2. Let $\nu = -\varepsilon'/\varepsilon$ denote the Poisson's ratio, where ε' is the vertical strain, find the vertical elongation δ' in terms of F, a, b, E, and ν.
3. Calculate the change in total area.

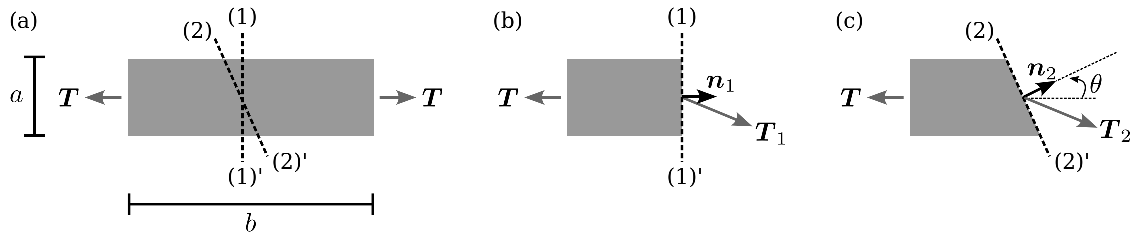

Figure 1: Two-dimensional bar problem. (a) A bar subjected to a uniform normal stress. (b) A freebody diagram with cross-section (1)-(1)'. (b) A freebody diagram with cross-section (2)-(2)'.

Problem 3 (Bar equation) We can approximate the general three-dimensional problem into a one-dimensional problem when considering a slender object subjected to an axial load. The corresponding one-dimensional problem is governed by the bar equation, i.e.,

$$\frac{d}{dx}\left[EA\frac{du}{dx}\right]+f=0, \quad 0<x<L. \tag{6}$$

In the above, E is the Young's modulus, A is the cross-sectional area, u is the axial displacement and f is the external force.

1. Consider the following boundary conditions:

$$u(0)=0 \quad \text{and} \tag{7}$$
$$u(L)=u_o. \tag{8}$$

   Here, $u_o>0$ is the given displacement at $x=L$. Let $f=0$ and EA is constant; find $u(x)$ by integrating the governing equation and applying the boundary condition.

2. In statics, we don't consider a case with homogeneous Neumann boundary conditions at both ends, i.e.,

$$EA\frac{du}{dx}=0 \quad \text{at} \quad x=0 \quad \text{and} \tag{9}$$
$$EA\frac{du}{dx}=0 \quad \text{at} \quad x=L. \tag{10}$$

   Provide mathematcial and physical interpretations why such boundary conditions are not used.