Final Exam - Mechanics of Materials

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Problem 1. (Hooke's law) The stress-strain relation for linear isotropic medium is given by

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{tr}\left(\boldsymbol{\sigma}\right)\boldsymbol{I}. \tag{1}$$

Here, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ is the strain tensor, E is the Young's modulus, ν is the Poisson's ratio, $\boldsymbol{I}$ is the identity matrix, and $\mathrm{tr}\left(\boldsymbol{\sigma}\right) = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$. Assume a pure normal stress, i.e.,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ \text{sym.} & & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma & 0 & 0 \\ & 0 & 0 \\ \text{sym.} & & 0 \end{bmatrix}. \tag{2}$$

Derive the Hooke's law:

$$\sigma = E\varepsilon, \tag{3}$$

where $\varepsilon = \varepsilon_{xx}$ is the $x - x$ component of the strain tensor.

Problem 2. (Principle of minimum potential energy) Consider a simply supported beam subjected to a point load at mid-span (Figure 1), where the point load can be modeled by a Dirac delta function, given by

$$q\left(x\right) = -P\delta\left(x - \frac{L}{2}\right). \tag{4}$$

Here, the negative sign indicates that the load acts downward. Then, the corresponding governing equation reads

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = -P\delta\left(x - \frac{L}{2}\right). \tag{5}$$

Suppose we are given with two trial functions:

$$w_1 = -\frac{PL}{16EI}x\left(L-x\right) \quad \text{and} \tag{6}$$

$$w_2 = -\frac{2PL^3}{\pi^4 EI}\sin\frac{\pi}{L}x. \tag{7}$$

(a) Compute the total potential energy $\Pi\left[w\right]$ for each trial function, where

$$\Pi\left[w\right] = \frac{1}{2}\int_0^L \frac{d^2w}{dx^2}EI\frac{d^2w}{dx^2}dx - \int_0^L qw dx. \tag{8}$$

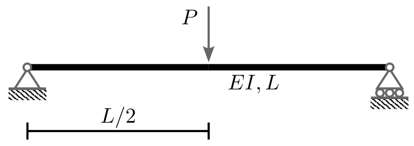

Figure 1: Simple beam.

(b) Using the principle of minimum potential energy, determine which of these trial functions provides a more accurate approximation. Note that $\pi^4 \approx 97.409$.

Hint: The Dirac delta function operates as follows with an arbitrary function $f(x)$

$$\int_0^L f(x)\,\delta(x - x_o)\,dx = f(x_o), \quad x_o \in (0, L). \tag{9}$$

Also, note the identity

$$\int_0^L \left(\sin\frac{\pi x}{L}\right)^2 dx = \frac{L}{2}. \tag{10}$$

Problem 3. (Buckling) Consider a fixed-free column subjected to an axial load p, as shown in Figure 2.

(a) State the boundary conditions at $x = 0$ for the lateral displacment $w(x)$.

(b) Assume that the lateral displacement takes the form of

$$w(x) = Ax^2, \tag{11}$$

where $A \in \mathbb{R}$ is an arbitrary number. Does the above choice satisfies the boundary conditions?

(c) Using the above assumption (11), calculate the total potential energy, i.e.,

$$\Pi[w] = \frac{1}{2}\int_0^L \frac{d^2w}{dx^2} EI \frac{d^2w}{dx^2} dx - \frac{1}{2}\int_0^L \frac{dw}{dx} p \frac{dw}{dx} dx. \tag{12}$$

(d) Sketch the total potential energy $\Pi[w]$ with respect to A for the following three cases: 1) $p < p_{\mathrm{cr}} = 3EI/L^2$, 2) $p = p_{\mathrm{cr}}$, and 3) $p > p_{\mathrm{cr}}$. Explain the result using the principle of minimum potential energy.

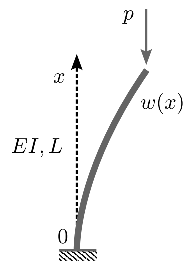

Figure 2: Fixed-free column.